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Goodness-of-fit tests for the Weibull distribution based on the Laplace transform

Titre: Tests d'adéquation à la loi de Weibull basés sur la transformée de Laplace

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Abstract: The aim of this paper is to develop new goodness-of-fit (GOF) tests for the two-parameter Weibull distribution based on the Laplace transform. The principle of the tests relies on the measure of the closeness between the theoretical Laplace transform and its empirical version. Three estimation methods are used to simplify the building of the statistics. The paper also introduces a new version of Cabaña and Quiroz statistic using the maximum likelihood estimators of the parameters. All these tests are not asymptotic and can be used for small samples size. A comprehensive comparison study is presented. Among all the proposed GOF tests, the best ones are identified. The results strongly depend on the shape of the underlying hazard rate.

Résumé : L'objectif de cet article est de développer de nouveaux tests d'adéquation à la loi de Weibull à deux paramètres basés sur la transformée de Laplace. Le principe de ces tests consiste à mesurer la proximité entre la transformée de Laplace théorique et sa version empirique. Trois méthodes d'estimation des paramètres de la loi de Weibull sont utilisées pour simplifier la construction des statistiques. L'article propose aussi une nouvelle version de la statistique de Cabaña et Quiroz utilisant les estimateurs de maximum de vraisemblance des paramètres. Ces tests ne sont pas asymptotiques, ils peuvent être utilisés pour des échantillons de petite taille. Une comparaison exhaustive des tests proposés est présentée. Parmi tous les tests d'adéquation utilisés, les meilleurs tests sont identifiés. Les résultats dépendent fortement de la forme du taux de hasard de la loi sous-jacente.

Keywords: Reliability, Goodness-of-fit tests, Weibull distribution, Extreme Value distribution, Laplace transform
Mots-clés : Fiabilité, Tests d'adéquation, Loi de Weibull, Loi des valeurs extrêmes, Transformée de Laplace
AMS 2000 subject classifications: 62N05, 62F03

1. Introduction

Risk management of industrial facilities needs to accurately predict system reliability. This requires, as a first step, the building of relevant probabilistic models in order to reflect the randomness of the occurrence of failures. In a second step, statistical inference of the developed models must be made, based on operation feedback data. A final step is firstly to validate the fitted models using statistical criteria and secondly to compare the different competing models.

Goodness-of-fit (GOF) tests are a useful tool to check the validity of models used in reliability. The exponential distribution is widely used in life testing, but it represents the disadvantage of having a constant failure rate. The Weibull distribution is a more flexible model since it allows decreasing, constant and increasing failure rates. It is then essential to be able to check its relevance for a given data set. There is a wide literature on GOF tests for the exponential distribution (Ascher, 1990; Henze and Meintanis, 2005), but GOF tests for the Weibull distribution have been much

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less studied (Mann et al., 1973; Tiku and Singh, 1981; Liao and Shimokawa, 1999; Cabaña and Quiroz, 2005). Reviews are presented in chapter 5 of Murthy et al. (2004) and chapter 22 of of Rinne (2009). Different approaches were used: tests based on regression, empirical distribution function, normalized spacings and likelihood based tests...

The use of the empirical Laplace transform or characteristic function in GOF testing has attracted a lot of attention (Baringhaus and Henze, 1991; Henze, 1993; Jimenez-Gamero et al., 2009; Henze et al., 2012; Meintanis et al., 2013). When testing exponentiality, the tests based on the Laplace transform seem to be among the most powerful ones (Baringhaus and Henze, 1991). The first study of GOF tests for the Weibull distribution based on the Laplace transform was done by Cabaña and Quiroz (2005). The principle of these tests relies on the measure of the closeness between the empirical Laplace transform and the theoretical one.

In this paper, we propose new GOF tests for the Weibull distribution mixing the ideas of Cabaña and Quiroz (2005) and those introduced by Henze (1993) for testing the exponential distribution.

The paper also introduces new versions of the two statistics of Cabaña and Quiroz using the maximum likelihood estimators instead of the moment estimators. The asymptotic convergence of the distribution of one of these statistics to the chi-squared distribution is established. The proposed tests are not asymptotic and can be applied to small samples. Finally a comprehensive comparison study is done.

Section 2 gives some important preliminary results. Section 3 is a reminder of both the works of Cabaña and Quiroz and of Henze. Section 4 details the building of the new statistics by combining both approaches and using three estimation methods of the parameters. We will detail in Section 5 the second version of Cabaña and Quiroz statistics using the maximum likelihood estimators and we prove the asymptotic convergence of the distribution of one of the statistics.

We conclude in Section 6 by a comparison of all the proposed GOF tests with some of the usual ones such as: Anderson-Darling (D'Agostino and Stephens, 1986), Tiku and Singh (1981), Mann-Scheuer and Fertig test (Mann et al., 1973) and Cabaña and Quiroz (2005).

2. Preliminary results

The two-parameter Weibull distribution $\mathscr{W}(\eta,\beta)$ is defined by its cumulative distribution function (cdf):

$$F(x;\eta,\beta) = 1 - e^{-(x/\eta)^{\beta}}, \quad x \geq 0,\, \eta > 0,\, \beta > 0. \tag{1}$$

Its probability density function (pdf) is:

$$f(x;\eta,\beta) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^{\beta}}, \quad x \geq 0,\, \eta > 0,\, \beta > 0. \tag{2}$$

When X is a random variable from the $\mathscr{W}(\eta,\beta)$ distribution, $\ln X$ has the type I extreme value distribution $\mathscr{EV}_1(\mu,\sigma)$ with cdf:

$$G(y;\mu,\sigma) = 1 - e^{-e^{(y-\mu)/\sigma}}, \quad y \in \mathbb{R},\, \mu \in \mathbb{R},\, \sigma > 0 \tag{3}$$

where $\mu = \ln\eta$ and $\sigma = 1/\beta$. Its pdf is:

$$g(y;\mu,\sigma) = \frac{1}{\sigma} e^{(y-\mu)/\sigma - e^{(y-\mu)/\sigma}}, \quad y \in \mathbb{R}. \tag{4}$$

$Y = \ln(X/\eta)^\beta = \beta(\ln X - \ln\eta) = (\ln X - \mu)/\sigma$ has the standard extreme value distribution $\mathscr{EV}_1(0,1)$, with pdf:

$$g(y;0,1) = e^{y-e^y}, \quad y \in \mathbb{R}. \tag{5}$$

Let $X_1,\ldots,X_n$ be n independent and identically distributed (i.i.d.) random variables with the $\mathscr{W}(\eta,\beta)$ distribution. The order statistics of this sample are denoted $X_1^* \leq \ldots \leq X_n^*$.

In this paper, we consider three methods for estimating the parameters η and β from the sample $X_1,\ldots,X_n$: the maximum likelihood, least squares and moment methods.

1. The maximum likelihood estimators (MLEs) of η and β, $\hat{\eta}_n$ and $\hat{\beta}_n$, are solutions of the following equations:

$$\hat{\eta}_n = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\hat{\beta}_n}\right)^{1/\hat{\beta}_n} \tag{6}$$

$$\frac{n}{\hat{\beta}_n} + \sum_{i=1}^{n} \ln X_i - \frac{n}{\sum_{i=1}^{n} X_i^{\hat{\beta}_n}} \sum_{i=1}^{n} X_i^{\hat{\beta}_n} \ln X_i = 0. \tag{7}$$

2. The Weibull probability plot (WPP) (Murthy et al., 2004) is the plot of points:

$$(\ln X_i^*, c_i),\, i \in \{1,\ldots,n\} \tag{8}$$

where $c_i = \ln[-\ln(1-p_i)]$ and p_i, $i \in \{1,\ldots,n\}$, are approximations of the order statistics of a uniform sample. Usual choices are symmetrical ranks $p_i = (i-0.5)/n$ and mean ranks $p_i = i/(n+1)$. In all that follows we use the symmetrical ranks. Under the Weibull assumption, these points should be approximately on a straight line (D'Agostino and Stephens, 1986).

The least squares estimators (LSEs) based on the WPP, $\widetilde{\eta}_n$ and $\widetilde{\beta}_n$, are defined as follows (Liao and Shimokawa, 1999):

$$\widetilde{\beta}_n = \frac{\sum_{i=1}^{n}(c_i - \overline{c})^2}{\sum_{i=1}^{n}(\ln X_i - \overline{\ln X})(c_i - \overline{c})} \tag{9}$$

$$\ln\widetilde{\eta}_n = \overline{\ln X} - \frac{\overline{c}}{\widetilde{\beta}_n} \tag{10}$$

where $\overline{\ln X} = \frac{1}{n}\sum_{i=1}^{n}\ln X_i$ and $\overline{c} = \frac{1}{n}\sum_{i=1}^{n} c_i$.

3. The moment estimators (MEs), $\breve{\eta}_n$ and $\breve{\beta}_n$, are defined as follows (Rinne, 2009):

$$\breve{\beta}_n = \frac{\pi}{\sqrt{6}}\left[\frac{1}{n-1}\sum_{i=1}^{n}(\ln X_i - \overline{\ln X})^2\right]^{-1/2} \tag{11}$$

$$\ln \breve{\eta}_n = \overline{\ln X} + \frac{\gamma}{\breve{\beta}_n} \tag{12}$$

where $\gamma = 0.577...$ is the Euler constant.

For all $i \in \{1,\dots,n\}$, $Y_i = \beta \ln(X_i/\eta)$ has the $\mathscr{E}\mathscr{V}_1(0,1)$ distribution. The order statistics of this sample are denoted $Y_1^* \leq \dots \leq Y_n^*$.

Since η and β are unknown, it will be useful in the following to replace them by the above estimators. For all i, let $\hat{Y}_i = \hat{\beta}_n \ln(X_i/\hat{\eta}_n)$, $\widetilde{Y}_i = \widetilde{\beta}_n \ln(X_i/\widetilde{\eta}_n)$ and $\breve{Y}_i = \breve{\beta}_n \ln(X_i/\breve{\eta}_n)$. It is expected that the distributions of $\hat{Y}_i$, $\widetilde{Y}_i$ and $\breve{Y}_i$ will not be far from the $\mathscr{E}\mathscr{V}_1(0,1)$ distribution.

From Antle and Bain (1969), the distribution of $(\hat{Y}_1,\dots,\hat{Y}_n)$ does not depend on η and β. From Liao and Shimokawa (1999), it is also the case of the distribution of $(\widetilde{Y}_1,\dots,\widetilde{Y}_n)$. The same result is proved for $(\breve{Y}_1,\dots,\breve{Y}_n)$ in Krit et al..

The fact that the distributions of the samples $\hat{Y}_i$, $\widetilde{Y}_i$ and $\breve{Y}_i$ are independent of the parameters of the underlying Weibull distribution is a fundamental property since it allows to build GOF test statistics as functions of these samples.

Let F be the cumulative distribution function of independent and identically distributed random variables $X_1,\dots,X_n$. A GOF test is a statistical test of hypothesis H_0: "$F \in \mathscr{F}$" vs H_1: "$F \notin \mathscr{F}$", where $\mathscr{F}$ is a family of distributions. In our case $\mathscr{F}$ will be the family of two-parameter Weibull distributions. If a positive random variable X has the Weibull distribution, then $\ln X$ has the type I extreme value distribution. So, thanks to a logarithmic transformation, a GOF test for the Weibull distribution is equivalent to a GOF test for the extreme value distribution.

3. GOF tests based on the Laplace transform: previous works

Henze (1993) proposed GOF tests for the exponential distribution based on the Laplace transform. The building of the test is based on the measure of the difference between the empirical Laplace transform and its theoretical version.

Let $X_1,\dots,X_n$ be independent and identically distributed (i.i.d) random variables from the exponential distribution with parameter λ, i.e with pdf $\lambda e^{-\lambda x}, x \geq 0$.

Let $Y_i = \lambda X_i$, $\forall i \in \{1,\dots,n\}$. $Y_1,\dots,Y_n$ is a sample from the unit exponential distribution. Its Laplace transform is:

$$\psi(t) = \mathbb{E}[e^{-tY_i}] = \frac{1}{1+t}. \tag{13}$$

Since λ is unknown it can be estimated by the maximum likelihood estimator $1/\overline{X}_n = n/\sum_{i=1}^n X_i$.

Let $\hat{Y}_i = X_i/\overline{X}_n, i \in \{1,\dots,n\}$. An important property is that the distribution of $\hat{Y}_1,\dots,\hat{Y}_n$ is independent of λ.

Henze's idea is to reject the hypothesis that $X_1,\dots,X_n$ are exponentially distributed if the empirical Laplace transform $\psi_n(t) = \frac{1}{n}\sum_{i=1}^n e^{-t\hat{Y}_i}$ is too far from the theoretical Laplace transform $\psi(t)$. The closeness between both functions is measured by a test statistic of the form:

$$n\int_0^{+\infty} |\psi_n(t) - 1/(1+t)|^2 w_a(t)dt \tag{14}$$

where $w_a(t) = e^{-at}$ is a weight function. Using the integration by parts, the test statistic turns out to be:

$$\frac{1}{n}\sum_{i,j=1}^{n}\frac{1}{Y_i+Y_j+a} - 2\sum_{j=1}^{n} e^{Y_j+a}E_1(Y_j+a) + n(1-ae^{a}E_1(a)) \tag{15}$$

where $E_1(z) = \int_z^{+\infty}\frac{e^{-t}}{t}dt$. The choice of the parameter a allows to build powerful GOF tests for a large range of alternatives.

Cabaña and Quiroz (2005) used the Laplace transform to build GOF tests for the Weibull and type I extreme value distributions. They exploited the fact that if $X_1,\ldots,X_n$ is an i.i.d sample from $\mathscr{W}(\eta,\beta)$ then the distribution of $Y_1,\ldots,Y_n$ follows the $\mathscr{EV}_1(0,1)$ distribution.

The Laplace transform of a sample $Y_1,...,Y_n$ from the $\mathscr{EV}_1(0,1)$ distribution is:

$$\psi(t) = \mathbb{E}\left[e^{-tY}\right] = \Gamma(1-t), \forall t<1$$

where Γ is the Gamma function defined as $\Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}\,\mathrm{d}x$.

Since the two parameters η and β are unknown, Cabaña and Quiroz proposed to replace Y_i by $\breve{Y}_i$, using the MEs of η and β defined in (11) and (12), and to use the empirical Laplace transform $\psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{-t\breve{Y}_j}$.

The closeness between the empirical and theoretical Laplace transform is measured by the empirical moment generating process

$$\breve{v}_n(s) = \sqrt{n}\Big(\frac{1}{n}\sum_{j=1}^{n} e^{-\breve{Y}_j s} - \Gamma(1-s)\Big).$$

Its distribution, under H_0, does not depend on the Weibull parameters η and β.

Cabaña and Quiroz proved the asymptotic convergence, under H_0, of $\breve{v}_n(s), s\in J = [-\delta^{'},\eta^{'}]$ for $\delta^{'}>0, \eta^{'}<0.5$ to a zero mean, continuous gaussian process $\breve{G}_p(s)$ indexed on J whose covariance matrix is completely known (Cabaña and Quiroz, 2005). In practice, Cabaña and Quiroz recommend to use $J=[-2.5,0.49]$ in order to keep the covariance matrix of the process away from singularity. They suggested two test statistics that are functions of the stochastic process $\breve{v}_n$. The first has the following quadratic form:

$$\breve{CQ}_n = \breve{v}_{n,S}V^{-1}(S)\,(\breve{v}_{n,S})^{\,t} \tag{16}$$

where $\breve{v}_{n,S} = (\breve{v}_n(s_1),\ldots,\breve{v}_n(s_k)), S=\{s_1,\ldots,s_k\}\subset J$ and $V(S)$ is the limiting covariance matrix of $\breve{v}_{n,S}$ given in equation (2.7) in Cabaña and Quiroz (2005). The statistic $\breve{CQ}_n$ has a limiting chi-squared distribution with k degrees of freedom. In the simulations presented in Section 6, we have chosen $k=2$, $s_1=-1$ and $s_2=0.4$ which are recommended in Cabaña and Quiroz (2005). The second test statistic is similar to the test of Henze: it is based on a weighted L^2 norm. The only difference being the choice of the weight function that is different from the one used by Henze:

$$\breve{S}_n = \int_J \breve{v}_n^2(s)/V(s)\,ds \tag{17}$$

where V is the limiting variance of $\breve{v}_n$. The asymptotic distribution of the test statistic $\breve{S}_n$ converges to the distribution of $\int_J \breve{G}_p^2(s)/V(s)\,ds$.

In the following we combine the two approaches, the one of Henze based on the weighted L^2 norm and the one based on Cabaña and Quiroz to build the new GOF tests by using the difference between the empirical Laplace transform of the transformed data $Y_i, i \in \{1,\ldots,n\}$, and the Laplace transform of the $\mathscr{EV}_1(0,1)$ distribution.

4. A new approach of the GOF tests building

Combining both approaches of Henze in (14) and of Cabaña and Quiroz in (17), we propose a test statistic of the following form:

$$n\int_I \Big(\frac{1}{n}\sum_{j=1}^{n} e^{-Y_j t} - \Gamma(1-t)\Big)^2 w_a(t)\,dt = \int_I v_n^2(t)\,w_a(t)\,dt \tag{18}$$

where w_a is a weight function and $I \subset]-\infty,1[$ is a bounded interval for which the above integral is convergent. The function w_a depends on a parameter a that can be chosen to obtain the best performance of the test as in Henze's work (Henze, 1993).

Henze chose $w_a(t) = e^{-at}$; this choice was justified by the fact of using a test of Cramer-Von-Mises type which gives an explicit expression of the statistics and a good power for different alternatives by adjusting the value of a. It is common in Cramer-Von-Mises and Anderson-Darling tests to use as a weight function the probability density function tested. Thus, we use as a weight function the probability density function of the $\mathscr{EV}_1(0,1)$ after dilatation with parameter a, $w_a(t) = e^{at-e^{at}}$.

For the exponential distribution, it was possible to find an explicit and simple expression of Henze's statistic as a function of the sample Y_j (see (15)). But, for the Weibull distribution, the integral (18) is not easy to compute since $\Gamma(1-t)$ is more complex than $\dfrac{1}{1+t}$. We can compute the integral using Simpson or Monte Carlo integration or we can simply compare the theoretical Laplace transform and the empirical one by discretizing the integral on an appropriately chosen interval I. For instance, with a discretization on $[0,1[$, we obtain the following test statistic:

$$LT_{a,m} = \sum_{k=1}^{m-1} v_n^2(k/m)w_a(k/m) = n\sum_{k=1}^{m-1}\Big[\frac{1}{n}\sum_{j=1}^{n} e^{-Y_j k/m} - \Gamma(1-k/m)\Big]^2 w_a(k/m). \tag{19}$$

The statistic $LT_{a,m}$ can be written as a quadratic form, as the first statistic of Cabaña and Quiroz:

$$LT_{a,m} = v_{n,m}W_a v_{n,m}^t \tag{20}$$

where $v_{n,m} = (v_n(\frac{1}{m}),\ldots,v_n(\frac{m-1}{m}))$ and $W_a = \begin{bmatrix} w_a(\frac{1}{m}) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & w_a(\frac{m-1}{m}) \end{bmatrix}$ is a diagonal weight matrix.

Equation (18) is similar to (17) and (20) is similar to (16), thus these tests are linked to those of Cabaña and Quiroz. But they are much simpler since they do not require the computation of the covariance matrix $V(s)$.

We tried different range values of t by discretizing the intervals $[-50,1[$, $[-10,1[$, $[-1,1[$, $[0,1[$, $[-1,0]$, $[-10,0]$ and $[-50,0]$. We used normalizing factors in some cases in order to have usual orders of magnitude of the statistics. The power results are similar for the statistics based on the discretizations of $[-50,1[$, $[-10,1[$, $[-1,1[$ and $[0,1[$. Similarly the statistics based on discretizing $[-1,0]$, $[-10,0]$ and $[-50,0]$ have a comparable performance. That is why we use only the discretizations of $[0,1[$ and $[-1,0]$. The two corresponding statistics are respectively denoted LT^1 and LT^2:

$$LT^1_{a,m} = n\sum_{k=1}^{m-1}\left[\frac{1}{n}\sum_{j=1}^{n} e^{-Y_j k/m} - \Gamma(1-k/m)\right]^2 w_a(k/m) \tag{21}$$

$$LT^2_{a,m} = n\sum_{k=-m}^{-1}\left[\frac{1}{n}\sum_{j=1}^{n} e^{-Y_j k/m} - \Gamma(1-k/m)\right]^2 w_a(k/m). \tag{22}$$

For a comparison purpose, let $LT^3_{a,m}$ be a third test statistic based on the discretization of the interval $[-2.5,0.49]$ recommended by Cabaña and Quiroz (2005):

$$LT^3_{a,m} = n\sum_{k=-2.5m}^{0.49m}\left[\frac{1}{n}\sum_{j=1}^{n} e^{-Y_j k/m} - \Gamma(1-k/m)\right]^2 w_a(k/m). \tag{23}$$

Each of the statistics (21), (22), (23) can be computed using $\hat{Y}_1,\dots,\hat{Y}_n$ or $\tilde{Y}_1,\dots,\tilde{Y}_n$ or $\check{Y}_1,\dots,\check{Y}_n$ instead of $Y_1,\dots,Y_n$. The corresponding statistics are denoted respectively $\widehat{LT}^i$, $\widetilde{LT}^i$ and $\check{LT}^i$, $i \in \{1,2,3\}$.
Using the moment estimators, we can conclude from the convergence result of $\check{v}_n(s), s \in J$ (Cabaña and Quiroz, 2005) and the continuous mapping theorem that $\check{LT}^i, i \in \{1,2,3\}$, converges under the null hypothesis H_0, to the distribution of:

$$\sum_{s\in I^i(m)} \check{G}^2_p(s) w_a(s)$$

where $I^1(m) = \{\frac{1}{m},\dots,\frac{m-1}{m}\}$, $I^2(m) = \{-1,\frac{-m+1}{m},\dots,\frac{-1}{m}\}$ and $I^3(m) = \{-2.5,\frac{-2.5m+1}{m},\dots,0.49\}$.

We have the same asymptotic convergence of the statistics $\widehat{LT}^i$ to $\sum_{s\in I^i(m)} \widehat{G}^2_p(s) w_a(s)$, where $\widehat{G}_p(s)$ is a zero mean continuous gaussian process with a specific covariance matrix that will be derived later in Section 5. Indeed, theorem 2.1 in Cabaña and Quiroz (2005) can be applied to the empirical process $\hat{v}_n$ using MLEs instead of the MEs, thus the sample $\hat{Y}_1,\dots,\hat{Y}_n$.

The behaviour of the tests statistics depends on the choice of the parameter value a of the weight function. It is impossible to find a value of the parameter a that maximizes the power of the GOF tests whatever the tested alternative. Indeed the behaviour of the tests depends in theory on the alternative tested and the sample size. After several simulations with different values of a, we recommend the use of $a = -5$ for both $\widehat{LT}^1_{a,m}$ and $\widehat{LT}^2_{a,m}$. We will use this value for the remaining tests statistics.

Concerning the choice of parameter m, it was fixed in all the simulations to $m = 100$. However $m = 100$ is not in all the cases the optimal m that gives the best performance. For instance, we studied the Monte Carlo estimation of the power of the test $LT^1_{-5,m}$ for a sample simulated from the Gamma distribution with parameters $(1,2)$. Figure 1 shows that the optimal value is $m = 70$ in

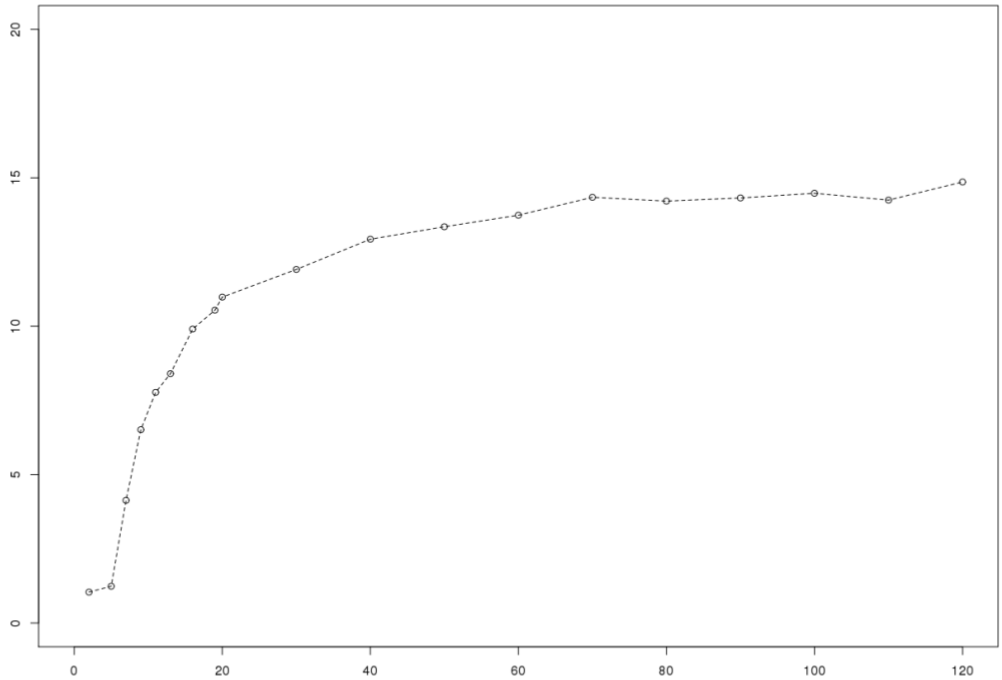

FIGURE 1. *The power of the test* $\widehat{LT}^1_{-5,m}$ *as a function of m*

this case. But this optimal value could have been different if we had simulated another distribution. Choosing a large value of m guarantees satisfying results in a large range of cases.

Given the expression of the new GOF test statistics as the distance between the theoretical and the empirical Laplace transforms, the null hypothesis H_0 is rejected when the statistics is too large. The Weibull assumption is rejected at the level α if the statistics is greater than the quantile of order $1-\alpha$ of its distribution under H_0. These quantiles are easily obtained by simulation.

5. Cabaña and Quiroz statistics with Maximum Likelihood Estimators

The results of Cabaña and Quiroz are valid for affine invariant estimators of $\mu = \ln \eta$ and $\sigma = \frac{1}{\beta}$ which are satisfying a condition denoted (2.6) in Cabaña and Quiroz (2005). Cabaña and Quiroz showed that this condition is fulfilled by the moment estimators, and obtained the tests statistics $\breve{CQ}_n$ and $\breve{S}_n$.

In this section, we prove that the MLEs verify the condition (2.6) in Cabaña and Quiroz (2005). So we are able to build the corresponding test statistics $\widehat{CQ}_n$ and $\widehat{S}_n$.

We know that the MLEs verify asymptotically the following condition (Theorem 5.39, page 65 Vaart, 1998):

$$\sqrt{n}\,(\hat{\mu}_n, \hat{\sigma}_n - 1)^t = \frac{1}{\sqrt{n}} I^{-1}_{(\mu=0,\sigma=1)} \sum_{i=1}^{n} \left(\frac{\partial \ln g}{\partial \mu}(Y_i, \mu = 0, \sigma = 1), \frac{\partial \ln g}{\partial \sigma}(Y_i, \mu = 0, \sigma = 1) \right)^t + o_p(1) \tag{24}$$

where I^{-1} is the inverse of the Fisher information matrix which can be derived as:

$$I^{-1}_{(\mu=0,\sigma=1)} = \begin{vmatrix} 1+\frac{6}{\pi^2}(1-\gamma)^2 & \frac{6}{\pi^2}(\gamma-1) \\ \frac{6}{\pi^2}(\gamma-1) & \frac{6}{\pi^2} \end{vmatrix}.$$

The condition becomes:

$$\sqrt{n}\,(\hat{\mu}_n, \hat{\sigma}_n - 1)^t = \frac{1}{\sqrt{n}} I^{-1}_{(\mu=0,\sigma=1)} \sum_{i=1}^{n} \left(-1+e^{Y_i}, -1-Y_i+Y_i e^{Y_i}\right)^t + o_p(1). \tag{25}$$

The two functions $K_1(y) = -1+e^y$ and $K_2(y) = -1-y+ye^y$ are linearly independent. Then, condition (2.6) in Cabaña and Quiroz (2005) is fulfilled for the MLEs and we can apply Theorem 2.1. Under the null hypothesis H_0, $\hat{v}_n = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} e^{-s\hat{Y}_j} - \Gamma(1-s)\right)$, as a stochastic process indexed on J, converges in distribution to a zero mean, sample continuous Gaussian process $\widehat{G}_p(s)$ with covariance structure given by:

$$\begin{aligned}
\mathbb{E}[\widehat{G}_p(v)\widehat{G}_p(u)] = {} & \Gamma(1-u-v) - \Gamma(1-u)\Gamma(1-v) \\
& + \bigtriangledown(v) I^{-1} \mathbb{E}\left[(-1+e^Y)e^{-uY}, (-1-Y+Ye^Y)e^{-uY}\right]^t \\
& + \bigtriangledown(u) I^{-1} \mathbb{E}\left[(-1+e^Y)e^{-vY}, (-1-Y+Ye^Y)e^{-vY}\right]^t \\
& + \bigtriangledown(u) I^{-1} Cov\left(-1+e^Y, -1-Y+Ye^Y\right)(I^{-1})^t \bigtriangledown(v)^t
\end{aligned}$$

where $\bigtriangledown(u) = u(-\Gamma(1-u), \Gamma'(1-u))$ and Y is a variable with the $\mathscr{E}\mathscr{V}_1(0,1)$ distribution.

After computation the limiting covariance structure is as follows:

$$\begin{aligned}
\mathbb{E}[\widehat{G}_p(v)\widehat{G}_p(u)] = {} & \Gamma(1-u-v) - \Gamma(1-u)\Gamma(1-v) \\
& + \bigtriangledown(v) I^{-1} \begin{pmatrix} \Gamma(2-u) - \Gamma(1-u) \\ -\Gamma(1-u) - \Gamma'(1-u) + \Gamma'(2-u) \end{pmatrix} \\
& + \bigtriangledown(u) I^{-1} \begin{pmatrix} \Gamma(2-v) - \Gamma(1-v) \\ -\Gamma(1-v) - \Gamma'(1-v) + \Gamma'(2-v) \end{pmatrix} \\
& + \bigtriangledown(u)(I^{-1})^t \bigtriangledown(v)^t.
\end{aligned}$$

We use the following results, using the change of variables $y = \ln t$:

$$\begin{aligned}
& \mathbb{E}[Ye^{-vY}] = \Gamma'(1-v) \\
& \mathbb{E}[Y^2 e^{-uY}] = \Gamma''(1-u) \\
& \mathbb{E}[(-1+e^Y)e^{-vY}] = \Gamma(2-v) - \Gamma(1-v) \\
& \mathbb{E}[(-1-Y+Ye^Y)e^{-uY}] = -\Gamma(1-u) - \Gamma'(1-u) + \Gamma'(2-u) \\
& \mathbb{V}ar(-1+e^Y) = 1 \\
& \mathbb{V}ar(-1-Y+Ye^Y) = \frac{\pi^2}{6} + (\gamma-1)^2 \\
& \mathbb{C}ov(-1+e^Y, -1-Y+Ye^Y) = 1-\gamma.
\end{aligned}$$

Hence, we can define new versions of the Cabaña and Quiroz statistics based on the MLEs instead of MEs:

$$\widehat{CQ}_n = \widehat{v}_{n,S}\widehat{V}^{-1}(S)\ (\widehat{v}_{n,S})^{\ t} \tag{26}$$

$$\hat{S}_n = \int_J \hat{v}_n^2(s)/\widehat{V}(s)\, ds \tag{27}$$

where $\widehat{v}_{n,S} = (\hat{v}_n(s_1), \ldots, \hat{v}_n(s_k)), S = \{s_1, \ldots, s_k\} \subset J$ and $\widehat{V}(S)$ is the limiting covariance matrix of $\hat{v}_{n,S}$ given above. The statistic $\widehat{CQ}_n$ has a limiting chi-squared distribution with k degrees of freedom. Figure 2 shows that the limiting variance of $\hat{v}_n$ grows very fast when s goes to $-\infty$ and the same when s approaches 0.5. In this case, we recommend that the interval J should be included in $[-1.5, 0.49]$. In the simulations presented in Section 6, we will use the test $\widehat{CQ}_n$ with the following values: $k = 2$, $s_1 = -0.1$ and $s_2 = 0.02$.

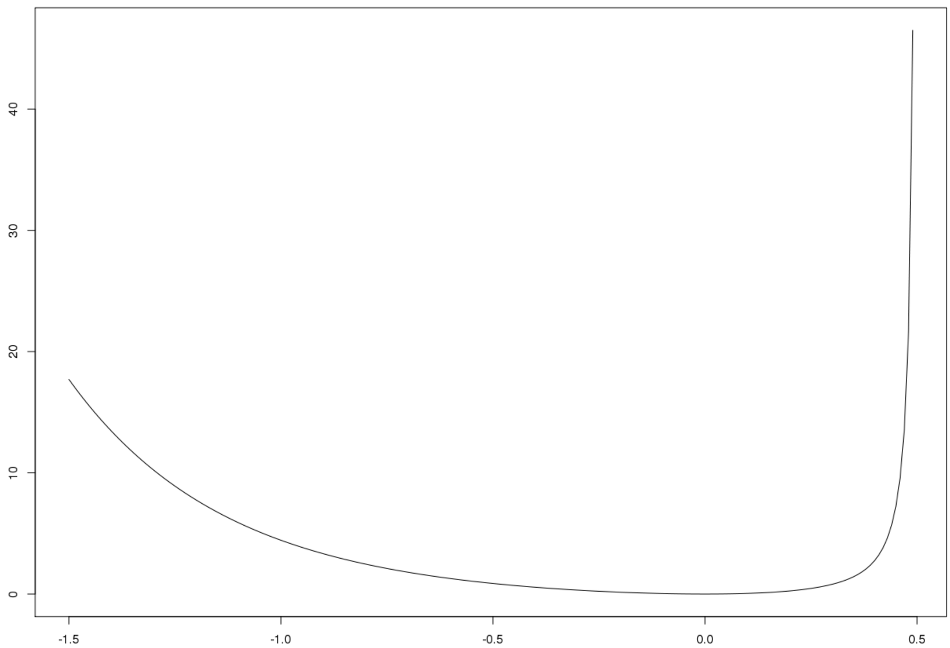

FIGURE 2. *The asymptotic variance of $\hat{v}_n(s)$ as a function of s*

Since the tests statistics are used for small values of the sample sizes, the asymptotic results are often not used in practice. That is why we had the idea of using a different version of the test statistic $\widehat{CQ}_n$ that we denote $\widehat{CQ}_n^*$, whose expression is $\widehat{CQ}_n^* = \widehat{v}_{n,S}\, A^{-1}\ (\widehat{v}_{n,S})^{\ t}$ by using any non singular matrix A. In this case, we have no more convergence of the test statistic distribution to a chi-squared distribution, but this is not important since we use simulated quantiles. Nevertheless we still have the property that the distribution of $\widehat{CQ}^*$ is independent of the parameters of the Weibull distribution under H_0. In the simulations in Section 6, we will use the test $\widehat{CQ}_n^*$ where $k = 2$, $S = \{-0.1, 0.02\}$ and we fix the following matrix $A = \begin{bmatrix} 1.59 & 0.91 \\ 0.91 & 0.53 \end{bmatrix}$.

6. Comparison of the goodness-of-fit tests

The previous section has proposed new GOF tests for the Weibull distribution. It is then important to select the best of them and compare them with the best GOF tests of the literature. The comparison of the proposed GOF tests will be based on the power of the tests. This section presents the results of an intensive Monte Carlo simulation study in order to assess the power of all the new GOF tests and to compare them with usual tests for the Weibull distribution.

The study is done using a broad range of alternative distributions. We have four classes depending on the shape of the hazard rates:

- IHR: increasing hazard rate
- DHR: decreasing hazard rate
- BT: bathtub-shaped hazard rate
- UBT: upside-down bathtub shaped hazard rate.

For each distribution, we simulate 50,000 samples of size $n \in \{10, 20, 50, 100\}$. All the GOF tests are applied with a significance level set to 5%. The tests reject Weibull hypothesis when the statistic is greater than the quantile of order 95% of its distribution under H_0. These quantiles are obtained by simulation, thus the asymptotic results are not used in this case.

The power of the tests is assessed by the percentage of rejection of the null hypothesis. The algorithms have been written in R.

We first simulate Weibull samples, in order to check that the percentage of rejection is close to the nominal significance level 5%. For the other simulations, we have chosen usual distributions such as:

- Gamma $\mathscr{G}$
- Lognormal $\mathscr{L}\mathscr{N}$
- Inverse-Gamma $\mathscr{I}\mathscr{G}$
- Generalized Weibull distributions:
  - Exponentiated Weibull distribution $\mathscr{E}\mathscr{W}(\theta, \eta, \beta)$ (Mudholkar and Srivastava, 1993) with the c.d.f:
$$F_X(x; \theta, \eta, \beta) = \left[1 - e^{-(x/\eta)^\beta}\right]^\theta$$
  - Generalized Gamma distribution $\mathscr{G}\mathscr{G}(k, \eta, \beta)$ (Stacy, 1962) with the c.d.f:
$$F_X(x; k, \eta, \beta) = \frac{1}{\Gamma(k)} \gamma(k, (x/\eta)^\beta)$$
    where $\gamma(s,x) = \int_0^x v^{s-1} e^{-v} dv$
  - Additive Weibull distribution $\mathscr{A}\mathscr{W}(\xi, \eta, \beta)$ (Xie and Lai, 1995; Bousquet et al., 2006) with the c.d.f:
$$F_X(x; \xi, \eta, \beta) = 1 - e^{-\xi x - (\frac{x}{\eta})^\beta}.$$

For the sake of simplicity, the scale parameters of the Weibull, Gamma and Inverse-Gamma distributions are set to 1 and the mean of the Lognormal distribution is set to 0. The choice of parameters of the simulated distributions is done in order to obtain different shapes of the hazard rate. Table 1 gives the values of the parameters and the notation used for all the simulated distributions.

We remind the values of the parameters used for the new test statistics:

TABLE 1. *Simulated distributions*

Weibull	$exp(1)$	$\mathscr{W}(1,0.5) \equiv \mathscr{W}(0.5)$	$\mathscr{W}(1,3) \equiv \mathscr{W}(3)$
IHR	$\mathscr{G}(2,1) \equiv \mathscr{G}(2)$	$\mathscr{G}(3,1) \equiv \mathscr{G}(3)$	$\mathscr{AW}1 \equiv \mathscr{AW}(10,0.02,5.2)$
DHR	$\mathscr{G}(0.2,1) \equiv \mathscr{G}(0.2)$	$\mathscr{AW}2 \equiv \mathscr{AW}(2,20,0.1)$	$\mathscr{EW}1 \equiv \mathscr{EW}(0.1,0.01,0.95)$
BT	$\mathscr{EW}2 \equiv \mathscr{EW}(0.1,100,5)$	$\mathscr{GG}1 \equiv \mathscr{GG}(0.1,1,4)$	$\mathscr{GG}2 \equiv \mathscr{GG}(0.2,1,3)$
UBT	$\mathscr{LN}(0,0.8) \equiv \mathscr{LN}(0.8)$	$\mathscr{LN}(0,2.4) \equiv \mathscr{LN}(2.4)$	$\mathscr{LN}(0,3) \equiv \mathscr{LN}(3)$
	$\mathscr{IG}(3,1) \equiv \mathscr{IG}(3)$	$\mathscr{GG}3 \equiv \mathscr{GG}(10,0.01,0.2)$	

– For $LT^i i \in \{1,2,3\}$: $m = 100$ and $a = -5$
– For $\check{CQ}$: $k = 2$, $S = \{-1, 0.4\}$
– For $\widehat{CQ}$: $k = 2$, $S = \{-0.1, 0.02\}$
– For $\widehat{CQ}^*$: $k = 2$, $S = \{-0.1, 0.02\}$ and $A = \begin{bmatrix} 1.59 & 0.91 \\ 0.91 & 0.53 \end{bmatrix}$.

For the power study, the percentage of rejection of H_0 is an estimation of the power of the test for this alternative. For instance, we see in Table 4 that the power of the $\widehat{LT}^1$ test for simulated $\mathscr{LN}(0,0.8)$ samples and $n = 20$ is estimated at 37.1%.

In the following tables, we assess the powers of the new GOF statistics $LT^i, i \in \{1,2,3\}$, with the three estimation methods and the new version of Cabaña and Quiroz test $\widehat{CQ}$.

We compare the performance of these new GOF tests to three usual GOF tests for the Weibull distribution and the one suggested by Cabaña and Quiroz:

– *AD*: Anderson Darling (D'Agostino and Stephens, 1986). The test statistic is:

$$AD = n\int_{-\infty}^{+\infty} \frac{\left[\mathbf{G}_n(y) - \hat{G}_0(y)\right]^2}{\hat{G}_0(y)(1-\hat{G}_0(y))}\, d\hat{G}_0(y) = -n + \frac{1}{n}\sum_{i=1}^{n}\left[(2i-1-2n)\ln(1-\hat{U}_i^*)) - (2i-1)\ln(\hat{U}_i^*)\right] \tag{28}$$

where $\mathbf{G}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\ln X_i \le x\}}$, $\hat{G}_0(y) = G(y;\ln\hat{\eta}_n, 1/\hat{\beta}_n)$ and $\hat{U}_i = \hat{G}_0(\ln X_i)$.

– *MSF*: Mann-Scheuer-Fertig (Mann et al., 1973). The test statistic is:

$$MSF = \frac{\sum_{j=\lfloor \frac{n}{2} \rfloor +1}^{n-1} E_j}{\sum_{j=1}^{n-1} E_j} \tag{29}$$

where $\lfloor x \rfloor$ is the integer part of x and $E_i = \frac{\ln X_{i+1}^* - \ln X_i^*}{\mathbb{E}[Y_{i+1}^*] - \mathbb{E}[Y_i^*]}$

– TS: Tiku and Singh (1981). The test statistic is:

$$TS = \frac{2\sum_{i=2}^{n-1}(n-i)E_i}{(n-2)\sum_{i=2}^{n}E_i} \tag{30}$$

– $\check{CQ}_n$: Cabaña and Quiroz (2005).

The last row of tables 2, 3, 4 and 5 gives the mean of rejection percentages of each test for all simulated alternative distributions, except the Weibull ones. This allows to identify the best tests for a broad range of alternatives.

TABLE 2. *Power results for the tests based on Laplace transform, $n = 100$*

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	TS	AD	MSF	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	5.1	5.1	4.9	5	5	5.3	5.1	5	5.1	5.1	4.9	5	5.1	5.1	5
$\mathscr{W}(0.5)$	5.1	5.1	4.9	4.9	4.9	5.1	5	5	4.9	5.1	4.9	5	4.9	5	4.8
$\mathscr{W}(3)$	5.3	5.1	5.1	5.1	5	5.2	5.1	5.1	4.9	5	4.9	4.9	5	5.1	5.1
$\mathscr{G}(2)$	22.1	17.1	2.2	23.2	39.5	6.8	17.9	10.8	9.2	18.8	13.3	21.5	22.8	11.3	19.6
$\mathscr{G}(3)$	38.4	31.7	5.7	27.5	39.5	12.3	28.9	21.6	17.1	34.9	23.3	34.6	40.4	23.9	34.7
$\mathscr{AW}1$	84.9	94.1	97.9	35.9	10.8	91.1	19.6	96	83	97.2	92.3	0	96.4	98.3	94.3
$\mathscr{G}(0.2)$	16.3	84.7	61.5	2.5	0.2	22.9	1.4	36.4	17.2	86.3	69.4	0	45.5	76.9	87.7
$\mathscr{AW}2$	60.7	100	99.7	6.8	0.3	66.2	2.8	86.1	36.3	100	100	0	98.9	100	100
$\mathscr{EW}1$	0	95.2	14.2	0.2	0.2	0	0.3	0.7	0	83.3	90.3	0	7.5	50.4	88.4
$\mathscr{EW}2$	0	95.3	14.4	0.1	0.2	0	0.3	0.7	0	83.1	90.4	0	7.7	50.4	88.5
$\mathscr{GG}1$	21.1	96.6	73.9	1.3	0.2	27.5	1.5	44.4	19.4	96.6	95	0	59.6	89.3	97.5
$\mathscr{GG}2$	16.1	84.4	61.2	2.5	0.2	22.7	1.4	36.8	16.7	85.7	78.2	0	45.6	77.2	88.2
$\mathscr{LN}(0.8)$	97.5	89.7	70.5	87.2	97.6	78.8	89.9	92.8	86	96.1	87.6	90.3	97.7	92.3	93.2
$\mathscr{LN}(2.4)$	97.6	89.9	70.9	87.4	97.7	78.5	89.8	92.5	86.2	96	87.8	90.6	97.8	92.5	93.1
$\mathscr{LN}(3)$	97.4	89.7	70.5	87.3	97.6	78.7	90	92.7	86.4	95.9	87.5	90.5	97.6	92.3	93.3
$\mathscr{IG}(3)$	100	99.7	99.2	99.9	100	99.9	100	100	100	100	99.8	99.8	100	99.9	99.9
$\mathscr{GG}3$	76.3	63.7	27.2	56.5	76.8	37.9	59.1	58.2	46.5	72.9	55	65.6	78.5	61.8	69.5
mean	52	80.8	54.9	37.1	40.1	44.5	35.9	55	43.2	81.9	76.4	28.1	64	72.6	82

TABLE 3. *Power results for the tests based on Laplace transform, $n = 50$*

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	TS	AD	MSF	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	4.9	5.3	5.1	5	4.9	4.8	4.8	5	4.9	4.9	5	5.1	5.1	4.8	5.1
$\mathscr{W}(0.5)$	5	5.2	5	5	5	5	4.9	5	4.9	5	5	5	5.1	5	5
$\mathscr{W}(3)$	5.1	5.2	5	5	4.9	5	4.8	5.2	5	5.1	5.2	5.1	5.1	5	5
$\mathscr{G}(2)$	14.5	11.7	1.4	13.4	15.6	7.6	13.6	6.7	10.2	11.9	8.5	14.4	15.4	4.7	11.6
$\mathscr{G}(3)$	23.2	18.2	1.4	19.8	24.2	11.9	20.2	10.7	15.8	18.9	13.2	21.6	25.1	8.4	18.6
$\mathscr{AW}1$	64.1	75.8	86.2	18.8	3.1	67.8	6	79.6	53.1	82.2	70.9	0.1	74.7	87.4	76.4
$\mathscr{G}(0.2)$	11.6	52.9	40.4	1.3	0.2	14.1	0.4	23.5	9.4	56.5	45.8	0.1	17.7	49.6	57.2
$\mathscr{AW}2$	44.8	99.9	93.1	3.6	0.3	41.7	0.5	61.1	20.3	99.6	99.9	0	74	98.3	99.9
$\mathscr{EW}1$	0.1	65.3	11.5	0.2	0.3	0.2	0.3	1.3	0.1	49.4	56.1	0	1.6	26.1	53.9
$\mathscr{EW}2$	0.1	65.3	11.2	0.2	0.3	0.2	0.3	1.3	0.1	49.8	56.2	0.1	1.6	26.5	54.4
$\mathscr{GG}1$	15	73.8	51.4	1.4	0.2	16.5	0.3	28.5	10.5	74.9	67.6	0	24.2	63.2	75.6
$\mathscr{GG}2$	11.8	53.1	40.9	1.2	0.2	13.9	0.3	23.4	9.3	56.2	45.5	0.1	17.9	49.9	57.1
$\mathscr{LN}(0.8)$	78.8	62.6	19.9	65.8	78.2	55.9	68.2	59.9	66	72	55.7	65.3	79.3	53.4	66.3
$\mathscr{LN}(2.4)$	78.7	62.4	20	65.8	78.4	56.3	67.8	60	66.2	72.8	55.3	65	79.4	52.7	65.7
$\mathscr{LN}(3)$	78.5	62.1	19.7	66.2	78.6	55.7	67.8	60.1	65.5	73	55.3	65.1	79.7	52.8	66
$\mathscr{IG}(3)$	98.6	91.3	66.7	96.7	98.8	93.1	97.5	95.5	97.3	96.9	91.6	93.3	98.3	91.2	93.5
$\mathscr{GG}3$	48.1	37.1	5.4	38.9	48.6	27.2	38.9	28.1	34.9	42.9	28.7	40.1	50.3	23.8	39.4
mean	40.6	59.4	33.5	28.1	30.5	33.1	27.3	38.6	32.8	61.2	53.6	26.1	45.6	49.2	59.7

TABLE 4. *Power results for the tests based on Laplace transform, $n = 20$*

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	TS	AD	MSF	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	4.9	4.9	4.9	5	5.1	5	4.9	5	5.1	5.1	5	5.1	5	5	5.4
$\mathscr{W}(0.5)$	4.8	4.9	5	5	5.1	5.1	4.9	5	5	5.2	4.9	5.2	4.9	5.2	5.3
$\mathscr{W}(3)$	4.9	5	5	5	5.1	4.9	5	5	4.9	5	5.1	5	5	5	5.5
$\mathscr{G}(2)$	9	7	1.9	9.5	10.1	8.5	9.3	3.8	9.7	6.6	5.9	9.5	10	2.8	6.6
$\mathscr{G}(3)$	12.1	8.7	1.1	12.7	13.6	11	4.6	0.9	13.3	8.7	7.3	12.5	5	2.4	8.4
$\mathscr{AW}1$	33.4	45.1	56.2	3.1	3.1	27.9	0.7	47.4	12.6	49.5	40.4	0.9	27.5	57.8	49.6
$\mathscr{G}(0.2)$	6.9	22.1	23.5	0.5	0.6	5	0.7	14.8	1.6	24.2	19.9	0.7	3.8	26.5	26
$\mathscr{AW}2$	25.9	87.9	66.7	0.5	0.8	16.4	0	37.9	4.1	87	88.7	0	19.4	75.4	87.1
$\mathscr{EW}1$	1	22.7	11	0.4	0.7	0.7	0.6	4.2	0.5	18.1	21.9	0.6	0.5	15.1	20.8
$\mathscr{EW}2$	0.9	22.8	10.8	0.4	0.7	0.7	0.6	3.9	0.5	17.6	21.7	0.7	0.4	14.8	21.3
$\mathscr{GG}1$	8.6	32.2	28.9	0.3	0.4	5.7	0.4	17.6	1.6	34.1	29.5	0.4	5	33.3	35.4
$\mathscr{GG}2$	6.8	22.6	23.6	0.5	0.6	5	0.6	14.8	1.6	24	20.1	0.7	3.9	26.7	25.9
$\mathscr{LN}(0.8)$	37.1	26.5	1.8	35.3	39.3	27.5	35.8	17.7	38.1	28.8	21.9	32.4	40.7	10.1	27.2
$\mathscr{LN}(2.4)$	37.3	26.6	1.8	35.3	39.7	27.1	35.5	17.7	38.2	29	21.9	31.8	40.5	9.7	26.7
$\mathscr{LN}(3)$	37.4	26.7	1.8	35.4	39.3	27	35.3	17.3	37.9	29.1	21.7	32.2	40.4	9.8	26.9
$\mathscr{IG}(3)$	68.9	51.6	10.5	67.7	71.9	45.3	68.6	47	71.4	59.7	48.3	58.7	70.3	31.1	53.3
$\mathscr{GG}3$	21.8	15.5	0.6	21.2	23.3	17	20.9	8.2	22.7	15.7	12	20	24.1	4.2	15.4
mean	21.9	29.9	17.2	15.9	17.4	16.1	15.3	18.1	18.1	30.8	27.2	14.4	21.4	22.8	30.8

TABLE 5. *Power results for the tests based on Laplace transform, $n = 10$*

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	TS	AD	MSF	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	5.1	5	4.9	4.9	5.1	5.4	5.2	5	4.8	4.9	5	4.8	4.9	5.1	5.2
$\mathscr{W}(0.5)$	5.4	5.1	4.8	5	5	5.2	5.1	4.8	4.9	5.2	5.3	5	5	5.1	5.1
$\mathscr{W}(3)$	5.1	5	5	5.1	5.2	5	5	4.9	5.1	5.1	5.1	5.1	5.1	5	5.2
$\mathscr{G}(2)$	7.6	5	2.6	7.8	7.9	7.4	7.8	3.1	7.7	5.6	5.1	7.4	7.8	2.7	4.5
$\mathscr{G}(3)$	9.3	5.5	1.8	9.3	9.4	8.2	9.1	2.6	9.3	6.2	5.4	9	9.6	1.9	4.7
$\mathscr{AW}1$	15.6	27.4	33.8	1.9	2.6	6.5	1.8	28.8	1.8	27.4	24.9	2	11.7	34.6	31.9
$\mathscr{G}(0.2)$	4	13.5	15.4	1.2	1.4	2.2	1.3	12.3	1.2	12	12.3	1.7	2.7	16.4	15.2
$\mathscr{AW}2$	14	56	43.3	0.1	0.9	3.6	0.2	29.6	0.1	51	54.7	0.3	9.2	47.5	53.9
$\mathscr{EW}1$	1.8	12.1	10.1	0.9	1.1	1.4	1.1	7.1	0.9	8.7	11.7	1.5	1.1	11.2	11.6
$\mathscr{EW}2$	1.9	12.1	10.1	1.1	1.4	1.4	1.2	7.3	1	8.9	12.1	1.5	1.2	11.3	11.8
$\mathscr{GG}1$	4.4	17.6	18.3	0.7	1	1.6	0.9	14.1	0.8	15.9	16.4	1.1	2.7	19.9	19.2
$\mathscr{GG}2$	4.1	13.4	15.5	1.2	1.3	2	1.4	12.3	1.2	11.8	12	1.6	2.7	16.3	15.2
$\mathscr{LN}(0.8)$	20.2	11	0.3	19.5	19.8	12.1	19.8	3.5	20.2	13.4	10	17.3	21	1	8.9
$\mathscr{LN}(2.4)$	20.4	11	0.3	19.7	20.1	12	19.7	3.4	20.2	13.5	10.2	17.2	20.3	0.9	9.2
$\mathscr{LN}(3)$	20.4	10.8	0.4	19.6	20	12.3	19.7	3.4	20.2	13.3	10.3	16.9	20.5	0.9	9
$\mathscr{IG}(3)$	37.3	21.9	0.2	35.7	37.2	13.8	37.4	9.4	37.8	27.5	21.1	31.2	37.1	0.9	19.2
$\mathscr{GG}3$	13.5	7.3	0.2	13.4	13.5	9.9	13.3	2.4	13.5	8.8	6.9	11.9	13.6	1.2	5.7
mean	12.5	16.1	10.9	9.4	9.8	6.7	9.6	9.9	9.7	16	15.2	8.6	11.5	12.1	15.7

6.1. Results and discussion

The first obvious result of the analysis of these tables is that the performance of the tests is strongly linked to the shape of the hazard rate of the simulated distribution. We see the same behaviour of the tests that appears for, on one hand the IHR and UBT alternatives and on the other hand the DHR and BT alternatives. This link is not surprising since an UBT hazard rate starts by increasing and a BT hazard rate starts by decreasing.

The second important remark is that the new GOF tests are biased for some alternatives except the test $\widehat{LT}^2$; their power is smaller than the significance level 5%. The same behaviour is noticed in Mann et al. (1973) for the Mann-Scheuer-Fertig test.

The tests based on the LSEs $\widetilde{LT}^i, i \in \{1,2\}$, the test based on the MEs $\check{LT}^1$ and the MSF test are powerful for IHR-UBT alternatives and biased for DHR-BT alternatives. The tests $\widehat{LT}^1, \widetilde{LT}^3, \check{LT}^i, i \in \{2,3\}$ and $\check{CQ}$ are biased for Exponentiated Weibull distributions ($\mathscr{EW}1$ and $\mathscr{EW}2$) for large n (≥ 20). For small values of the sample size $n \leq 10$, the tests $\widehat{LT}^1$, $\widetilde{LT}^i, i \in \{1,2,3\}$, $\check{LT}^i, i \in \{2,3\}$, and $\check{CQ}$ are biased for the DHR-BT alternatives and the tests $\widehat{LT}^3$ and $\widehat{CQ}$ become biased for IHR-UBT alternatives (except for the alternative $\mathscr{AW}1$ for $n = 10$).

The two tests $\widehat{CQ}$ and $\check{CQ}$ depend on the choice of the values of S. The test $\widehat{CQ}_n^*$ depends on both the value of S and the choice of the matrix A, we may have better performances for different values than those used for the comparison. The tables comparison shows that the test $\widehat{CQ}$ is more powerful than $\check{CQ}$, but the results can be very different depending on the choice of S. The test $\widehat{CQ}^*$ is the most powerful among both $\check{CQ}$ and $\widehat{CQ}$: its performance is very close and competitive with TS.

The only non biased test for all the sample sizes is the test based on the MLEs $\widehat{LT}^2$. It has a good performance compared to the tests TS and AD.

The performance of the test statistics is very dependent on the shape of the hazard rate. The

GOF tests have the following behaviour:

- For the IHR alternatives: $\widetilde{LT}^2$ is more powerful than TS except for the alternative $\mathscr{AW}1$ where the power is less than all the powers of the new GOF tests. The performance of the test $\widehat{LT}^2$ is very close to that of TS test.
- For the DHR-BT alternatives: the new GOF tests based on the LSEs $\widetilde{LT}^i, i \in \{1,2\}$, and the MEs $\check{LT}^1$ are biased for all the DHR-BT alternatives. These tests have the same performance as the MSF test. The two tests $\widetilde{LT}^3$ and $\check{LT}^3$ become biased for small values of $n \leq 20$ not only for the alternatives $\mathscr{EW}i$, $i = 1,2$, but for all the remaining DHR-BT alternatives.
- For the UBT alternatives: the three tests $\widehat{LT}^1$, $\widetilde{LT}^2$ and $\check{CQ}$ are very powerful, even the most powerful compared to the usual tests including TS. For $n \leq 20$, the test $\check{CQ}$ becomes much more powerful than TS for the UBT alternatives and loses the performance it has against DHR-BT alternatives and becomes biased in this case.

For the majority of the studied alternatives, there exists a new GOF test that is significantly more powerful than the usual tests but, no test is uniformly better than the usual ones. Globally, the two best tests among all the new GOF tests are $\widehat{LT}^2$ and $\widehat{CQ}^*$ which are more powerful than MSF and AD and have good performances comparable to TS.

6.2. Conclusion and future work

In this paper, we introduced new goodness-of-fit tests for the Weibull distribution based on the Laplace transform. Each one of the new GOF tests has three versions depending on the estimation methods used. The advantage of these new GOF tests is the fact that they are exact and can be applied to any sample size.

In a previous study we showed that the powers of Weibull GOF tests are significantly lower than the ones of the exponential GOF tests (Krit and Gaudoin, 2012). For small sample sizes, the powers are quite low for all tests.

The performance of the tests depends strongly on the shape of the hazard rate of the underlying distribution. The new tests can be very interesting when the shape of the hazard rate is known. Nonparametric estimation of the hazard rate can be done before choosing which test to use. In this case, we recommend to use $\widehat{LT}^2$ for DHR-BT alternatives, $\check{CQ}$ for UBT alternatives and $\widehat{LT}^1$ for IHR alternatives. The tests based on the least squares and the moment estimators have to be used carefully because they are biased. It is safer to use the unbiased test $\widehat{LT}^2$ that has as good performance as Tiku-Singh test TS.

Since the results of some test statistics seem to be complementary, one of the prospects for future work is to combine these statistics using multiple testing in order to have better performances whatever the monotony of the hazard rate. The second prospect is to adapt these tests for censored data after introducing the censoring in the estimation of the MLEs. Another prospect is to study the impact of the weight function to the test performances.

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