Journal de la Société Française de Statistique
Vol. 154 No. 1 (2013)

A regularized goodness-of-fit test for copulas

Titre: Un test d'adéquation de copule régularisé

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Abstract: The authors propose an Anderson–Darling-type statistic for copula goodness-of-fit testing. They determine the asymptotic distribution of the statistic under the null hypothesis. As this distribution depends on the unknown value of the copula parameter, they call on a multiplier method to compute the p-value of the test. They assess the power of the test through simulations and find that it is generally superior to that of the Cramér–von Mises statistic based on the distance between the empirical copula and a consistent parametric copula estimate under $\mathscr{H}_0$.

Résumé : Les auteurs proposent une statistique de type Anderson–Darling pour tester l'ajustement d'une copule. Ils déterminent la loi limite de la statistique sous l'hypothèse nulle. Puisque cette loi dépend de la valeur inconnue du paramètre de la copule, ils font appel à une approche par multiplicateurs pour le calcul du seuil du test. Ils évaluent la puissance du test par voie de simulation et trouvent qu'elle surpasse généralement celle du test de Cramér–von Mises fondé sur la distance entre la copule empirique et une estimation paramétrique de la copule convergente sous $\mathscr{H}_0$.

Keywords: Anderson–Darling statistic, Cramér–von Mises statistic, empirical copula, Gaussian process, Monte Carlo study, Multiplier Central Limit Theorem, pseudo-observation, rank.
Mots-clés : statistique de Anderson–Darling, statistique de Cramér–von Mises, copule empirique, processus Gaussien, étude de Monte Carlo, théorème central limit à multiplicateurs, pseudo-observation, rang.
AMS 2000 subject classifications: 62H15, 62G20, 60G15

1. Introduction

Let $X_1,\ldots,X_d$ be $d \geq 2$ continuous random variables with cumulative distribution functions $F_1,\ldots,F_d$, respectively. As is well known (see, e.g., [9, 23]), the dependence among the components of $\mathbf{X} = (X_1,\ldots,X_d)$ is then characterized by a copula C, i.e., the joint cumulative distribution function of the transformed vector $\mathbf{U} = (U_1,\ldots,U_d) = (F_1(X_1),\ldots,F_d(X_d))$.

Given $n \geq 2$ mutually independent copies $\mathbf{X}_1 = (X_{11},\ldots,X_{1d}),\ldots,\mathbf{X}_n = (X_{n1},\ldots,X_{nd})$ of $\mathbf{X}$, it is of interest to test the hypothesis $\mathscr{H}_0 : C \in \mathscr{C}_0$ that the unique underlying copula C belongs to a class $\mathscr{C}_0 = \{C_\theta : \theta \in \mathscr{O}\}$ parameterized by θ ranging in some open set $\mathscr{O}$. Goodness-of-fit testing procedures for copulas were reviewed in [1, 13]; more recent contributions include [18, 19].

Conceptually at least, the simplest goodness-of-fit test for copulas [12] consists in comparing the distance between a nonparametric estimate $\hat{C}_n$ of C and a parametric estimate C_{θ_n} derived

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from an estimator θ_n of θ which is consistent when $\mathscr{H}_0$ holds. This approach relies on the ranks of the observations, except in the rare instances where $X_1,\ldots,X_d$ have known distributions.

Let $F_{n1},\ldots,F_{nd}$ be the empirical distribution functions of $F_1,\ldots,F_d$, respectively. For each $i \in \{1,\ldots,n\}$, a natural substitute for the unobservable $\mathbf{U}_i = (F_1(X_{i1}),\ldots,F_d(X_{id}))$ is given by

$$\hat{\mathbf{U}}_i = (\hat{U}_{i1},\ldots,\hat{U}_{id}) = \left(\frac{n}{n+1}F_{n1}(X_{i1}),\ldots,\frac{n}{n+1}F_{nd}(X_{id})\right),$$

so that for arbitrary $j \in \{1,\ldots,d\}$, $(n+1)\hat{U}_{ij}$ is the rank of X_{ij} among $X_{1j},\ldots,X_{nj}$. A natural estimator $\hat{C}_n$ of C, called the empirical copula [9], is then defined, for all $u_1,\ldots,u_d \in [0,1]$, by

$$\hat{C}_n(u_1,\ldots,u_d) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(\hat{U}_{i1} \le u_1,\ldots,\hat{U}_{id} \le u_d).$$

It is known to estimate C consistently and to have a weak limit under minimal regularity conditions [6, 7, 26, 27, 28]. The scaling factor $n/(n+1)$ used in defining $\hat{\mathbf{U}}_i$ avoids numerical issues that sometimes occur, e.g., when a parametric copula density is evaluated at pseudo-observations lying on the frontier of $[0,1]^d$. This rescaling has no effect on the limiting behaviour of C_n.

When $\mathscr{H}_0$ holds, various estimates of the copula parameter θ can be derived from the pseudo-sample $\hat{\mathbf{U}}_1,\ldots,\hat{\mathbf{U}}_n$. When C_θ admits a density c_θ, a standard solution called maximum pseudo-likelihood consists in finding the value θ_n of θ that maximizes

$$\ell(\theta) = \sum_{i=1}^{n}\ln\{c_\theta(\hat{U}_{i1},\ldots,\hat{U}_{id})\}.$$

As shown in [10], this estimator is asymptotically Gaussian and unbiased under weak regularity conditions. It is also known to be nearly efficient and robust to misspecification of the margins [15, 16]. When θ is real-valued, moment-based techniques such as inversion of Kendall's tau, Spearman's rho or Blomqvist's beta are also possible; see, e.g., [8, 16].

From extensive numerical studies [1, 13, 19], it appears that overall, the "blanket" statistic

$$\begin{aligned} S_n &= n\int_{[0,1]^d}\{\hat{C}_n(u_1,\ldots,u_d) - C_{\theta_n}(u_1,\ldots,u_d)\}^2 \mathrm{d}\hat{C}_n(u_1,\ldots,u_d) \\ &= \sum_{i=1}^{n}\{\hat{C}_n(\hat{U}_{i1},\ldots,\hat{U}_{id}) - C_{\theta_n}(\hat{U}_{i1},\ldots,\hat{U}_{id})\}^2 \end{aligned}$$

based on the L_2 norm between $\hat{C}_n$ and C_{θ_n} provides an excellent combination of power and simplicity, as compared to alternative statistics based, e.g., on the L_∞ norm or Rosenblatt transforms. This procedure, proposed in [12], assigns equal weight to departures between $\hat{C}_n$ and C_{θ_n} at all pseudo-observations $\hat{\mathbf{U}}_1,\ldots,\hat{\mathbf{U}}_n$. In many applications, however, it is important to ensure that the copula dependence structure fits well in the tails, i.e., when C_{θ_n} is either close to 0 or 1. This is the case, e.g., in economics, finance, and risk management, where the joint occurrence of low or high extremes can have disastrous consequences; see, e.g., [21].

This consideration leads naturally to Anderson–Darling-type statistics such as

$$R_n = n\int_{[0,1]^d}\left[\frac{\hat{C}_n(u_1,\ldots,u_d) - C_{\theta_n}(u_1,\ldots,u_d)}{[C_{\theta_n}(u_1,\ldots,u_d)\{1 - C_{\theta_n}(u_1,\ldots,u_d)\} + \zeta_m]^m}\right]^2 \mathrm{d}\hat{C}_n(u_1,\ldots,u_d),$$

involving a consistent, rank-based estimator θ_n of θ, and tuning parameters $m \geq 0$ and $\zeta_m \geq 0$. Given that $\hat{C}_n$ is a discrete measure, R_n is just as simple to compute as S_n; in fact, R_n coincides with S_n when $m = 0$. However, setting $m > 0$ makes it possible to emphasize discrepancies between $\hat{C}_n$ and C_{θ_n} in the tails, i.e., at points $\hat{\mathbf{U}}_1, \ldots, \hat{\mathbf{U}}_n$ where $C_{\theta_n}(1 - C_{\theta_n})$ is close to zero. At the same time, setting $\zeta_m > 0$ avoids the denominator getting too close to zero, which might spawn both theoretical and practical difficulties; see [5] for an illustration in a different context.

The purpose of this paper is to investigate the small- and large-sample properties of a goodness-of-fit test for copulas based on the statistic R_n. Its asymptotic null distribution is given in Section 2, where the test that rejects $\mathscr{H}_0$ for large values of R_n is also shown to be consistent.

As is typical of goodness-of-fit test statistics for copulas, the asymptotic null distribution of R_n depends on the unknown value of the underlying copula parameter θ. Therefore, one must resort to a resampling algorithm to compute p-values. While a parametric bootstrap could be used to this end [12], a more recent technique based on the Multiplier Central Limit Theorem [3, 19, 25] turns out to be much more efficient. This technique is described in Section 3 in the special case where the copula parameter θ is estimated by inversion of Kendall's tau.

The results of a modest simulation study reported in Section 4 show that when $m = .5$ and $\zeta_m = .05$, the power of the test based on R_n is comparable to, and often considerably higher than, the power of the test based on S_n, which corresponds to the case $m = 0$. The problem of how best to choose the tuning parameters m and ζ_m in order to maximize the power of R_n is not addressed here; this issue and others are listed in the Conclusion as possible avenues for future research.

2. Asymptotic null distribution of the test statistic

The large-sample distribution of the statistic R_n depends on the joint behaviour, as $n \to \infty$, of the sequence $\sqrt{n}(\hat{\theta}_n - \theta)$ and the empirical copula process $\sqrt{n}(\hat{C}_n - C)$. The weak convergence of the latter has been thoroughly investigated; see, e.g., [6, 7, 26, 28]. Let $\mathbb{B}_C$ be a centred Gaussian field on $[0,1]^d$ with covariance structure given, for all $u_1, \ldots, u_d, v_1, \ldots, v_d \in [0,1]$, by

$$\mathrm{cov}\{\mathbb{B}_C(u_1, \ldots, u_d), \mathbb{B}_C(v_1, \ldots, v_d)\} = C(u_1 \wedge v_1, \ldots, u_d \wedge v_d) - C(u_1, \ldots, u_d)C(v_1, \ldots, v_d),$$

where in general, $a \wedge b = \min(a,b)$. Further let $\ell^\infty[0,1]^d$ be the set of bounded functions from $[0,1]^d$ to $\mathbb{R}$, endowed with the uniform norm $\|\cdot\|$. The following result is taken from [27].

Lemma 1. *Suppose that for all $j \in \{1, \ldots, d\}$, $\dot{C}_j(u_1, \ldots, u_d) = \partial C(u_1, \ldots, u_d)/\partial u_j$ is continuous on $\mathscr{L}_j = \{(u_1, \ldots, u_d) \in [0,1]^d : u_j \in (0,1)\}$. Then as $n \to \infty$, $\sqrt{n}(\hat{C}_n - C)$ converges weakly on $\ell^\infty[0,1]^d$ to a tucked Brownian bridge $\mathbb{C}$ defined, for all $u_1, \ldots, u_d \in [0,1]$, by*

$$\mathbb{C}(u_1, \ldots, u_d) = \mathbb{B}_C(u_1, \ldots, u_d) - \sum_{j=1}^{d} \dot{C}_j(u_1, \ldots, u_d)\mathbb{B}_C(1, \ldots, 1, u_j, 1, \ldots, 1).$$

Next, assume that C belongs to an identifiable family $\mathscr{C}_0 = \{C_\theta : \theta \in \mathscr{O}\}$ of copulas, where $\mathscr{O} \subset \mathbb{R}^p$ is open. Further assume that the following conditions hold for all $\theta \in \mathscr{O}$:

$\mathscr{A}_1$: For all $j \in \{1, \ldots, d\}$, $\partial C_\theta(u_1, \ldots, u_d)/\partial u_j$ is continuous on $\mathscr{L}_j$.
$\mathscr{A}_2$: As $n \to \infty$, $\sqrt{n}(\hat{C}_n - C_{\theta_n}, \theta_n - \theta)$ converges weakly to a Gaussian limit $(\mathbb{C}_\theta, \Theta)$ in $\ell^\infty[0,1]^d \otimes \mathbb{R}^p$.

$\mathscr{A}_3$: The gradient vector $\nabla C = \partial C/\partial\theta$ exists and, as $\varepsilon \downarrow 0$,

$$\sup_{\|\theta^*-\theta\|<\varepsilon} \|\nabla C_{\theta^*} - \nabla C_\theta\| \to 0.$$

These conditions are satisfied for a large number of copula families and for various rank-based techniques commonly used for estimation purposes in this context [9]. The following lemma is a trivial extension of a result stated in [2, 19] for the case $\theta \in \mathbb{R}$.

Lemma 2. *If conditions $\mathscr{A}_1 - \mathscr{A}_3$ hold, then, as $n \to \infty$, $\sqrt{n}(\hat{C}_n - C_{\theta_n})$ converges weakly on $\ell^\infty[0,1]^d$ to the tight centred Gaussian process defined, for all $u_1,\ldots,u_d \in [0,1]$, by*

$$\mathbb{C}_\theta(u_1,\ldots,u_d) - \Theta^\top \nabla C_\theta(u_1,\ldots,u_d).$$

As proved in the Appendix, the limiting null distribution of the statistic R_n is the following.

Proposition 1. *Under $\mathscr{H}_0$ and conditions $\mathscr{A}_1 - \mathscr{A}_3$, and as $n \to \infty$, R_n converges weakly to*

$$R = \int_{[0,1]^d} \left[\frac{\mathbb{C}_\theta(u_1,\ldots,u_d) - \Theta^\top \nabla C_\theta(u_1,\ldots,u_d)}{[C_\theta(u_1,\ldots,u_d)\{1 - C_\theta(u_1,\ldots,u_d)\} + \zeta_m]^m} \right]^2 \mathrm{d}C_\theta(u_1,\ldots,u_d).$$

Now assume that $\mathscr{H}_0$ does not hold and that condition $\mathscr{A}_2$ is replaced by

$\mathscr{A}_2'$: There exists $\theta^* \in \mathscr{O}$ such that, as $n \to \infty$, $\sqrt{n}(\hat{C}_n - C, \theta_n - \theta^*)$ converges to a Gaussian limit $(\mathbb{C}_\theta, \Theta)$ in $\ell^\infty[0,1]^d \otimes \mathbb{R}^p$.

The following result then implies that the test of $\mathscr{H}_0$ based on R_n is consistent.

Proposition 2. *Suppose that conditions $\mathscr{A}_1$, $\mathscr{A}_2'$ and $\mathscr{A}_3$ are satisfied while $\mathscr{H}_0$ does not hold. Then, for arbitrary $\varepsilon > 0$, $\Pr(R_n > \varepsilon) \to 1$ as $n \to \infty$.*

3. Validity of the multiplier method

It is clear from Proposition 1 that the limiting null distribution of the statistic R_n depends on the unknown value of the copula parameter θ. To determine whether the observed value r_n of R_n is sufficiently large to reject $\mathscr{H}_0$, one must thus approximate the p-value $\Pr(R_n > r_n|\mathscr{H}_0)$ via a numerically intensive method, e.g., a parametric bootstrap [12] or a multiplier method [3, 19, 25]. The latter technique is preferred here as it proves to be computationally more efficient.

As simulations presented in Section 4 involve only bivariate copulas with real-valued parameter θ, the following description of the multiplier method is restricted to that case. In order to construct multiplier replicates of the statistic R_n, estimates $\dot{C}_{1n}$ and $\dot{C}_{2n}$ of the partial derivatives $\dot{C}_1$ and $\dot{C}_2$ are needed. Following [14, 25], let $\ell_n \in (0,1/2)$ be a bandwidth parameter such that

$$\lim_{n\to\infty} \ell_n = 0, \quad \inf\{\sqrt{n}\ell_n > 0 : n \in \mathbb{N}\} > 0.$$

In practice, $\ell_n = 1/\sqrt{n}$ seems an effective choice. For arbitrary $u_1, u_2 \in [0,1]$, set

$$\dot{C}_{1n}(u_1,u_2) = \frac{\hat{C}_n(u_1+\ell_n,u_2)-\hat{C}_n(u_1-\ell_n,u_2)}{2\ell_n},$$
$$\dot{C}_{2n}(u_1,u_2) = \frac{\hat{C}_n(u_1,u_2+\ell_n)-\hat{C}_n(u_1,u_2-\ell_n)}{2\ell_n}.$$

Proceeding as in [19], let M be a large integer and, for each $h \in \{1,\ldots,M\}$, let $Z^{(h)} = (Z_1^{(h)},\ldots,Z_n^{(h)})$ be a vector of mutually independent random variables with mean 0 and variance 1. These variables, which must also be completely independent from the data, are taken as $\mathcal{N}(0,1)$ in what follows. Further write $\bar{Z}^{(h)} = (Z_1^{(h)}+\cdots+Z_n^{(h)})/n$ and, for all $u_1,u_2 \in [0,1]$, set

$$\mathbb{C}_n^{(h)}(u_1,u_2) = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_i^{(h)}-\bar{Z}^{(h)})\mathbf{1}(\hat{U}_{i1}\le u_1, \hat{U}_{i2}\le u_2)$$

and $\hat{\mathbb{C}}_n^{(h)}(u_1,u_2) = \mathbb{C}_n^{(h)}(u_1,u_2) - \dot{C}_{1n}(u_1,u_2)\,\mathbb{C}_n^{(h)}(u_1,1) - \dot{C}_{2n}(u_1,u_2)\,\mathbb{C}_n^{(h)}(1,u_2)$. These are bootstrap replicates of the empirical copula process $\hat{\mathbb{C}}_n$.

Next, the choice of estimator for θ must be taken into account. While the maximum pseudo-likelihood estimation technique would be the method of choice and there is ample empirical evidence that the multiplier method is valid in this case [19], a formal proof is still lacking. For this reason, the following presentation focuses on moment-based methods.

In many bivariate copula families, the dependence parameter θ is typically in one-to-one correspondence with Kendall's tau, defined by

$$\tau(\theta) = -1+4\int_0^1\int_0^1 C_\theta(u_1,u_2)\mathrm{d}C_\theta(u_1,u_2).$$

An estimate θ_n of θ is then given by $\theta_n = \tau^{-1}(\tau_n)$, where τ_n is the empirical version of Kendall's tau. This estimator is generally consistent [12]. Furthermore, it can be expressed in the form

$$\sqrt{n}(\theta_n-\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^n J_\theta\{F_1X_{i1},F_2(X_{i2})\} \tag{1}$$

for an appropriate choice of score function J_θ. Assuming that $\tau'(\theta) = \partial\tau(\theta)/\partial\theta$ exists everywhere on $\mathscr{O}$, one can actually check [2] that, for all $u_1,u_2 \in [0,1]$, by

$$J_\theta(u_1,u_2) = \frac{4}{\tau'(\theta)}\left\{2C_\theta(u_1,u_2)-u_1-u_2+\frac{1-\tau(\theta)}{2}\right\}.$$

Accordingly, if one sets

$$\Theta^{(h)} = \frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i^{(h)} J_{\theta_n}(\hat{U}_{i1},\hat{U}_{i2}),$$

a multiplier replicate $R_n^{(h)}$ of R_n is given, for each $h \in \{1,\ldots,M\}$, by

$$R_n^{(h)} = \int_{[0,1]^2}\left[\frac{\hat{\mathbb{C}}_n^{(h)}(u_1,u_2)-\Theta^{(h)}\nabla C_{\theta_n}(u_1,u_2)}{[C_{\theta_n}(u_1,u_2)\{1-C_{\theta_n}(u_1,u_2)\}+\zeta_m]^m}\right]^2 \mathrm{d}\hat{C}_n(u_1,u_2),$$

where ∇C_θ simply stands for $\partial C/\partial\theta$ given that $\theta \in \mathbb{R}$, which is assumed to exist. An approximate p-value for the test that $\mathscr{H}_0 : C \in \mathscr{C}_0$ based on the statistic R_n is then given by

$$p = \frac{1}{M}\sum_{h=1}^{M} \mathbf{1}(R_n^{(h)} \geq r_n).$$

The following result ensures the reliability of this resampling approach. Its proof, given in the Appendix, relies on Theorem 3 of [19], which imposes various regularity conditions on J_θ. Estimators of the form (1) that meet these conditions are said to belong to the class $\mathscr{R}_2$.

Proposition 3. *Suppose that conditions $\mathscr{A}_1 - \mathscr{A}_3$ hold and that an estimator θ_n of θ belongs to the class $\mathscr{R}_2$ of rank-based estimators as defined by Kojadinovic et al.* [19]. *Then, as $n \to \infty$, $(R_n, R_n^{(1)}, \ldots, R_n^{(M)}) \rightsquigarrow (R, R^{(1)}, \ldots, R^{(M)})$, where $\rightsquigarrow$ denotes weak convergence, R is as in Proposition 1, and $R, R^{(1)}, \ldots, R^{(M)}$ are mutually independent and identically distributed.*

A similar result holds *mutatis mutandis* for estimators of θ that belong to the class $\mathscr{R}_1$ of rank-based estimators considered in [19]. These are estimators that are expressible in the form

$$\sqrt{n}\,(\theta_n - \theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} J_\theta(\hat{U}_{i1}, \hat{U}_{i2}) + o_p(1)$$

in terms of a score function J_θ satisfying more stringent regularity conditions. Its proof is very similar to that of Proposition 3 but relies on Theorem 2 of [19] instead. The estimator $\theta_n = \rho^{-1}(\rho_n)$ of θ obtained by inversion of Spearman's rho is of the latter type; the corresponding score function is defined, for all $u_1, u_2 \in [0,1]$, by $J_\theta(u_1,u_2) = \{12u_1u_2 - 3 - \rho(\theta)\}/\rho'(\theta)$.

4. Power study

In order to assess the finite-sample performance of R_n as a goodness-of-fit test statistic, a power study was conducted. To facilitate comparisons, simulations were carried out according to the same experimental design as in [13, 19]. Each of the following six copula families served both as the true and hypothesized dependence model: the Clayton, Frank, Gumbel, Normal, Plackett, and Student t_4 copula with four degrees of freedom. Three levels of dependence were considered, corresponding to population values of .25, .5 and .75 for Kendall's tau. Simulations were carried out for various sample sizes and estimation techniques, as well as for a broad selection of values for m and ζ_m. Source codes based on the R package `copula` [17] are available upon request.

For the sake of brevity, the results presented below are limited to three sample sizes and two configurations of tuning parameters, viz., $n \in \{150, 300, 500\}$, $m \in \{0, .5\}$, and $\zeta_m = .05$. When $m = 0$, the value of ζ_m is irrelevant, as R_n then reduces to the Cramér–von Mises statistic S_n. All calculations are based on 1000 replicates; in each case, the null distribution was approximated from $M = 1000$ multipliers using standard Normal variates $Z^{(1)}, \ldots, Z^{(M)}$. Furthermore, the bandwidth $\ell_n = 1/\sqrt{n}$ was used to compute $\dot{C}_{1n}$ and $\dot{C}_{2n}$. As already mentioned in Section 3, the copula parameter θ was estimated throughout by inversion of Kendall's tau. Conclusions for other rank-based estimators were similar.

Tables 1–3 show the results for n = 150, 300 and 500, respectively. In these tables, the underlined characters refer to the observed level of the tests. To facilitate comparisons between

TABLE 1. *Percentage of rejection of $\mathscr{H}_0$ at the 5% nominal level in 1000 goodness-of-fit tests based on statistics R_n ($m = .5, \zeta_m = .05$) and S_n ($m = 0$) for samples of size $n = 150$ when $M = 1000$ multipliers are used.*

	R_n						S_n					
$\tau = .25$	Copula under $\mathscr{H}_0$											
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	4.9	**71.1**	**88.8**	**58.3**	**47.8**	**66.9**	4.5	45.3	74.9	38.8	38.0	43.3
F	*53.2*	5.1	17.5	*4.4*	6.9	4.1	60.0	4.6	16.9	7.1	9.8	4.3
G	82.4	*30.3*	5.7	*20.6*	20.3	*27.5*	85.9	35.9	5.0	27.0	24.3	33.2
N	*42.4*	11.0	**19.2**	4.3	4.6	8.5	50.6	7.8	12.6	4.0	5.1	7.0
t_4	*45.4*	**31.6**	**30.9**	12.4	5.7	**23.6**	53.6	17.2	16.0	8.9	4.9	13.0
P	*47.9*	6.3	19.6	5.2	*5.0*	5.2	55.3	5.6	16.5	6.0	7.9	5.0
$\tau = .5$												
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	3.8	**98.9**	99.9	**94.0**	**90.3**	**96.9**	3.6	92.3	99.7	85.2	84.6	87.9
F	96.0	5.2	*30.8*	*11.3*	*19.8*	3.6	95.1	3.7	36.7	17.5	25.1	4.7
G	100.0	74.0	5.2	45.2	39.6	58.6	100.0	69.1	4.7	43.6	38.3	53.9
N	93.8	**40.7**	**29.9**	4.0	3.6	**17.2**	93.3	23.1	19.2	3.9	4.2	10.3
t_4	95.2	**65.8**	**43.9**	**13.3**	5.7	**33.2**	94.3	42.2	25.0	7.5	4.3	16.3
P	95.2	**15.8**	31.7	*6.0*	*7.9*	3.7	95.0	9.0	30.6	9.8	11.6	3.6
$\tau = .75$												
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	4.3	**100.0**	100.0	**96.4**	**95.4**	**97.9**	3.4	96.8	100.0	92.6	91.4	90.0
F	99.4	3.6	*34.3*	*27.3*	*37.8*	5.6	98.7	2.5	44.8	36.6	45.1	8.5
G	100.0	**97.0**	4.4	**51.6**	**43.7**	**69.4**	100.0	84.5	3.6	38.5	35.6	48.5
N	99.7	**83.7**	**20.1**	3.2	2.1	**18.0**	99.4	45.4	11.0	2.3	2.5	5.4
t_4	99.9	**94.3**	**31.5**	**8.8**	3.7	**32.3**	99.3	62.4	16.2	4.1	2.3	12.3
P	99.7	**51.0**	24.1	*8.3*	*6.7*	4.6	99.5	24.4	24.4	11.9	11.9	3.5

TABLE 2. *Percentage of rejection of $\mathscr{H}_0$ at the 5% nominal level in 1000 goodness-of-fit tests based on statistics R_n ($m = .5, \zeta_m = .05$) and S_n ($m = 0$) for samples of size $n = 300$ when $M = 1000$ multipliers are used.*

	R_n						S_n					
$\tau = .25$	Copula under $\mathscr{H}_0$											
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	4.8	**93.2**	**99.4**	**84.4**	**77.0**	**90.5**	5.2	77.8	97.8	69.3	71.9	76.0
F	85.0	4.9	35.5	7.6	23.0	4.6	83.8	4.1	36.8	10.4	23.9	4.3
G	98.5	*46.7*	4.6	*32.8*	36.8	*41.9*	98.8	53.0	4.2	39.6	36.1	48.0
N	76.7	15.0	**29.7**	5.0	7.7	12.2	78.3	11.8	24.0	4.5	8.3	9.8
t_4	*74.2*	**47.3**	**47.8**	**18.6**	4.6	**35.4**	79.2	30.6	26.7	11.8	4.4	20.7
P	82.5	5.7	33.2	7.0	13.9	4.9	83.0	6.1	32.1	9.5	15.6	5.2

$\tau = .5$												
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	5.6	100.0	100.0	99.9	99.9	100.0	6.0	100.0	100.0	99.7	99.7	100.0
F	100.0	3.8	76.4	*33.2*	61.7	6.3	100.0	3.1	80.8	38.9	59.6	7.8
G	100.0	94.5	4.5	69.4	66.8	**84.0**	100.0	92.3	4.1	66.3	61.7	78.7
N	100.0	**63.0**	**54.5**	3.9	4.3	**26.4**	100.0	46.0	42.6	3.7	4.7	18.9
t_4	99.9	**90.3**	**69.0**	**15.6**	4.4	**52.3**	99.9	75.3	47.9	9.3	3.4	31.1
P	100.0	**21.3**	62.3	*15.8*	23.1	5.3	100.0	15.0	61.7	21.0	26.3	5.7

$\tau = .75$												
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	2.7	100.0	100.0	100.0	100.0	100.0	2.3	100.0	100.0	100.0	100.0	100.0
F	100.0	3.7	91.1	79.8	91.6	23.7	100.0	2.7	91.5	79.9	89.2	26.0
G	100.0	100.0	3.4	**74.8**	**70.1**	**92.3**	100.0	99.4	3.1	63.4	59.3	78.9
N	100.0	**99.2**	**48.0**	3.1	2.4	34.8	100.0	84.7	31.9	2.7	2.7	16.6
t_4	100.0	**100.0**	**60.3**	**8.6**	3.0	**53.5**	100.0	96.1	36.8	4.9	2.7	22.8
P	100.0	**70.9**	57.9	*16.6*	20.3	3.1	100.0	45.0	56.8	23.0	24.8	2.5

TABLE 3. *Percentage of rejection of $\mathscr{H}_0$ at the 5% nominal level in 1000 goodness-of-fit tests based on statistics R_n ($m = .5, \zeta_m = .05$) and S_n ($m = 0$) for samples of size $n = 500$ when $M = 1000$ multipliers are used.*

$\tau = .25$	R_n						S_n					
	Copula under $\mathscr{H}_0$											
True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	4.1	**99.6**	100.0	**97.7**	95.5	**99.2**	3.4	96.0	100.0	92.3	94.3	95.1
F	97.0	4.2	59.7	12.0	46.0	4.1	95.8	4.4	63.5	14.4	41.9	4.2
G	100.0	*71.5*	4.9	55.3	**64.6**	*66.4*	100.0	77.3	4.4	58.2	57.8	72.1
N	93.5	**20.9**	**47.3**	4.8	16.0	15.4	92.7	15.7	41.4	4.6	15.0	12.0
t_4	94.1	**69.6**	**69.4**	**28.3**	5.8	**52.8**	95.6	52.4	47.6	19.4	5.2	36.9
P	96.9	4.9	59.4	*8.5*	31.6	2.8	95.8	4.7	60.6	12.4	29.9	3.8

$\tau = .5$ True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	4.0	100.0	100.0	100.0	100.0	100.0	4.4	100.0	100.0	100.0	100.0	100.0
F	100.0	4.0	97.5	65.0	91.6	11.4	100.0	4.0	98.4	64.6	88.3	11.8
G	100.0	99.7	4.8	88.8	**88.6**	97.3	100.0	99.2	4.1	86.6	83.9	96.2
N	100.0	**83.5**	**79.7**	4.1	8.0	**43.1**	100.0	69.4	69.6	4.0	8.0	31.7
t_4	100.0	**98.4**	**88.9**	**21.1**	4.6	**71.3**	100.0	92.1	75.1	12.7	4.4	52.5
P	100.0	**30.1**	92.6	31.7	49.4	4.5	100.0	20.3	91.7	36.7	48.1	4.6

$\tau = .75$ True copula	C	F	G	N	t_4	P	C	F	G	N	t_4	P
C	2.4	100.0	100.0	100.0	100.0	100.0	2.3	100.0	100.0	100.0	100.0	100.0
F	100.0	3.1	99.9	**98.9**	99.9	57.2	100.0	2.3	99.9	97.3	99.6	52.8
G	100.0	100.0	3.3	**94.3**	**91.8**	**99.6**	100.0	100.0	3.0	86.9	84.9	96.7
N	100.0	**100.0**	**80.6**	3.2	3.1	**60.1**	100.0	98.8	61.9	2.4	3.6	35.0
t_4	100.0	100.0	**87.2**	**9.1**	3.0	**77.5**	100.0	99.9	67.7	5.7	2.6	42.7
P	100.0	**91.0**	91.5	*40.7*	50.7	3.6	100.0	70.1	90.2	49.9	51.3	2.6

the power $\Pi(R_n)$ of the test based on R_n and the power $\Pi(S_n)$ of the test based on S_n, their difference was tested at the 1% level. Differences that are significant at that level are typeset in **boldface** if R_n is superior to S_n, and in *italics* in the reverse case.

The following observations can be made:

(a) Both tests maintain their level fairly closely when $\tau = .25$ or .5, but less so when $\tau = .75$. In all cases, the observed level tends to be closer to its nominal value with increasing sample size. This is consistent with the observations reported in [19] for the case of S_n.

(b) $\Pi(R_n)$ is generally superior to $\Pi(S_n)$. The observed counts of (**bold**, *italics*, black) cells are (**37**, *19*, 52), (**29**, *7*, 72) and (**30**, *4*, 74) in Tables 1, 2, and 3, respectively. Generally speaking, the superiority of R_n over S_n becomes more apparent as τ and n increase.

(c) As reported elsewhere [1, 13, 19], the Archimedean copulas (Clayton, Frank, and Gumbel) are more easily distinguished than the other three. Both R_n and S_n achieve their lowest power when trying to tell apart the two meta-elliptical copulas (Normal and t_4).

The tables also reflect the asymmetric roles of $\hat{C}_n$ and C_{θ_n} in the definitions of R_n and S_n; e.g., when $n = 150$ and $\tau = .5$, R_n rejects the Frank copula in 74% of cases when the data are Gumbel, while the latter is rejected in only 30.8% of cases when the data are Frank. The corresponding figures for S_n are 69.1 and 36.7, respectively.

5. Concluding remarks

The purpose of this paper was to propose a new, rank-based goodness-of-fit test procedure for copulas. The test is based on an Anderson–Darling-type statistic R_n which emphasizes the observed differences between the empirical copula $\hat{C}_n$ and a parametric copula C_{θ_n} of the true copula C derived from a consistent, rank-based estimator θ_n of the dependence parameter θ under the hypothesis $\mathscr{H}_0 : C \in \mathscr{C}_0 = \{C_\theta : \theta \in \mathscr{O}\}$, where $\mathscr{O}$ is some open set.

In view of the simulations, the test based on the regularized statistic R_n is a strong competitor to the Cramér–von Mises test statistic S_n. The dependence of R_n on the tuning parameters m and $\zeta_m > 0$ is open to debate, however. The choice $m = .5$ used here is natural, given that the numerator and denominator in the integrand are then on the same scale, but other options might improve the power of the test against specific alternatives. It is more difficult to defend the choice $\zeta_m = .05$ made here by trial and error. While the power of the test appears relatively unaffected by this tuning parameter, one could conceivably dispose of it with some extra theoretical work. Variants of the Anderson–Darling statistic considered here could also be envisaged. For instance, one might replace C_{θ_n} by $\hat{C}_n$ in the definition of R_n. While this change would not alter the limiting null distribution of the statistic, it could affect the power of the test in small samples.

Finally, the present developments suppose that the data used for model fitting and validation form a random sample. This is typically not the case in econometric or financial applications, where copulas are used to model the dependence between residuals from multivariate time series; see [24] for a review. With some additional effort, recent work on the asymptotic behaviour of the empirical copula process under serial dependence could be used to extend results in this direction; see, e.g., [4] and references therein.

Acknowledgments

Thanks are due to Prof. Johanna Nešlehová (McGill University) and Dr Marius Hofert (ETH Zürich) for many stimulating discussions and some technical assistance. This work was supported by the Canada Research Chairs Program, the Canadian Network of Centres of Excellence, the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, the Fonds de recherche du Québec (Nature et Technologies, Société et Culture), the Institut de finance mathématique de Montréal, and the William Dow Chair in Political Economy (McGill University).

Appendix: Proofs

The proof of Proposition 1 relies on the following result, which may be of independent interest; a related result is Proposition 7.27 in [20], although a multivariate extension thereof would be needed here. In what follows, $\xrightarrow{p}$ refers to convergence in probability.

Proposition 4. *Let C be a d-variate copula and $\hat{C}_n$ be the corresponding empirical copula based on a random sample of size n. If a sequence $\mathbb{G}_n$ of processes is tight with respect to the uniform norm on the space $\mathscr{C}[0,1]^d$ of continuous functions on $[0,1]^d$, then, as $n \to \infty$,*

$$\Delta_n = \int_{[0,1]^d} \mathbb{G}_n \mathrm{d}C_n - \int_{[0,1]^d} \mathbb{G}_n \mathrm{d}C \xrightarrow{p} 0.$$

Proof. By hypothesis, for any $m \in \mathbb{N}$, there exists a compact subset K_m of $\mathscr{C}[0,1]^d$ such that $\Pr(\mathbb{G}_n \notin K_m) < 1/m$ for all $n \in \mathbb{N}$. Therefore, the proof is complete if one can show that for arbitrary $\varepsilon > 0$, $\Pr\{\mathbb{G}_n \in K_m, |\Delta_n| > \varepsilon\} \to 0$ as $n \to \infty$.

Given that K_m is compact, it is totally bounded, so for any given $\delta > 0$, one can find a finite covering of K_m by balls $B_1, \ldots, B_L$ of radius $\delta > 0$, where for $\ell \in \{1, \ldots, L\}$, B_ℓ is centred at $g_\ell \in \mathscr{C}[0,1]^d$. As K_m is a normal space, it also follows from Theorem 5.1 in Chapter 4 of [22] that there exist continuous positive mappings $\phi_1, \ldots, \phi_L$ such that for each $\ell \in \{1, \ldots, L\}$, the support of ϕ_ℓ is contained in B_ℓ and, for every $g \in K_m$, $\phi_1(g) + \cdots + \phi_L(g) = 1$.

Thus, on the event $\{\mathbb{G}_n \in K_m\}$,

$$\begin{aligned} \Delta_n &= \sum_{\ell=1}^{L} \phi_\ell(\mathbb{G}_n) \int_{[0,1]^d} \mathbb{G}_n \mathrm{d}\hat{C}_n - \sum_{\ell=1}^{L} \phi_\ell(\mathbb{G}_n) \int_{[0,1]^d} \mathbb{G}_n \mathrm{d}C \\ &= \sum_{\ell=1}^{L} \phi_\ell(\mathbb{G}_n) \int_{[0,1]^d} (\mathbb{G}_n - g_\ell) \mathrm{d}\hat{C}_n - \sum_{\ell=1}^{L} \phi_\ell(\mathbb{G}_n) \int_{[0,1]^d} (\mathbb{G}_n - g_\ell) \mathrm{d}C \\ &\quad + \sum_{\ell=1}^{L} \phi_\ell(\mathbb{G}_n) \left(\int_{[0,1]^d} g_\ell \, \mathrm{d}\hat{C}_n - \int_{[0,1]^d} g_\ell \, \mathrm{d}C \right). \end{aligned}$$

It follows that on the event $\{\mathbb{G}_n \in K_m\}$, one has

$$|\Delta_n| \le 2\delta + \max_{\ell \in \{1,\ldots,L\}} \left| \int_{[0,1]^d} g_\ell \, \mathrm{d}\hat{C}_n - \int_{[0,1]^d} g_\ell \, \mathrm{d}C \right|.$$

Choosing $\delta = \varepsilon/4$, one then gets

$$\Pr\{\mathbb{G}_n \in K_m, |\Delta_n| > \varepsilon\} \leq \Pr\left\{\max_{\ell\in\{1,\ldots,L\}} \left|\int_{[0,1]^d} g_\ell \,\mathrm{d}\hat{C}_n - \int_{[0,1]^d} g_\ell \,\mathrm{d}C\right| > \varepsilon/2\right\}.$$

To see that the latter term tends to zero asymptotically, it suffices to note that, as $n \to \infty$,

$$\int_{[0,1]^d} g_\ell \,\mathrm{d}\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} g_\ell\left(\hat{U}_{i1},\ldots,\hat{U}_{id}\right) \xrightarrow{p} \int_{[0,1]^d} g_\ell \,\mathrm{d}C.$$

In fact, given that g_ℓ is continuous and bounded, the convergence occurs almost surely. In dimension $d = 2$, this follows, e.g., from part (i) of Proposition A·1 in [10]; when the function is bounded, however, the same proof therein carries over to arbitrary dimension $d \geq 2$. □

Proof of Proposition 1: First observe that conditions $\mathscr{A}_1 - \mathscr{A}_3$ imply that, as $n \to \infty$, $C_{\theta_n} \xrightarrow{p} C_\theta$. Indeed, for fixed $\varepsilon, \delta > 0$, one has

$$\Pr(\|C_{\theta_n} - C_\theta\| > \varepsilon) \leq \Pr(\|\theta_n - \theta\| \geq \delta) + \Pr(\|C_{\theta_n} - C_\theta\| > \varepsilon, \|\theta_n - \theta\| < \delta)$$

and, as $n \to \infty$, the first summand goes to 0 because the estimator θ_n is consistent. Furthermore,

$$\Pr(\|C_{\theta_n} - C_\theta\| > \varepsilon, \|\theta_n - \theta\| < \delta) \leq \Pr\left(\sup_{\|\theta^* - \theta\| < \delta} \|C_{\theta^*} - C_\theta\| > \varepsilon\right)$$

and the right-hand term vanishes asymptotically by condition $\mathscr{A}_3$.

Now define $D_\theta = \{C_\theta(1-C_\theta) + \zeta_m\}^{2m}$. Given that $|s(1-s) - t(1-t)| \leq 3|s-t|$ for all $s,t \in [0,1]$, it follows from the Continuous Mapping Theorem that, as $n \to \infty$,

$$\|D_{\theta_n} - D_\theta\| \xrightarrow{p} 0. \tag{A1}$$

Next consider the mapping $\psi_\theta : \mathscr{C}[0,1]^d \to \mathscr{C}[0,1]^d : g \mapsto \psi_\theta(g) = g^2/D_\theta$. Given that $\|D_\theta\| \geq \zeta_m^{2m}$, it follows that for all $g,h \in \mathscr{C}[0,1]^d$, $\|\psi_\theta(g) - \psi_\theta(h)\| \leq \|g^2 - h^2\|/\zeta_m^{2m}$, whence ψ_θ is continuous. Lemma 2 and the Continuous Mapping Theorem then jointly imply that, as $n \to \infty$,

$$\psi_\theta\{\sqrt{n}(\hat{C}_n - C_{\theta_n})\} = \frac{n(\hat{C}_n - C_{\theta_n})^2}{D_\theta} \rightsquigarrow \frac{(\mathbb{C}_\theta - \Theta^\top \nabla C_\theta)^2}{D_\theta} = \psi_\theta(\mathbb{C}_\theta - \Theta^\top \nabla C_\theta) \tag{A2}$$

in $\ell^\infty[0,1]^d$. Furthermore, note that

$$\left\|\frac{n(\hat{C}_n - C_{\theta_n})^2}{D_\theta} - \frac{n(\hat{C}_n - C_{\theta_n})^2}{D_{\theta_n}}\right\| \leq \|n(\hat{C}_n - C_{\theta_n})^2\| \times \frac{\|D_{\theta_n} - D_\theta\|}{\zeta_m^{4m}}.$$

It thus follows from (A1) and the tightness of the process $\sqrt{n}(\hat{C}_n - C_{\theta_n})$ that, as $n \to \infty$,

$$\left\|\frac{n(\hat{C}_n - C_{\theta_n})^2}{D_\theta} - \frac{n(\hat{C}_n - C_{\theta_n})^2}{D_{\theta_n}}\right\| \xrightarrow{p} 0.$$

Combining this fact with (A2), one then has, as $n \to \infty$,

$$\frac{n(\hat{C}_n - C_{\theta_n})^2}{D_{\theta_n}} \rightsquigarrow \frac{(\mathbb{C}_\theta - \Theta^\top \nabla C_\theta)^2}{D_\theta}.$$

The conclusion now follows from Proposition 4 and the Continuous Mapping Theorem. □

Proof of Proposition 2: First decompose the goodness-of-fit process in the form

$$\sqrt{n}\,(\hat{C}_n - C_{\theta_n}) = \sqrt{n}\,(\hat{C}_n - C) - \sqrt{n}\,(C_{\theta_n} - C_{\theta^*}) + \sqrt{n}\,(C - C_{\theta^*}).$$

A simple adaptation of the proof of Lemma 2 then implies that, as $n \to \infty$, $\sqrt{n}\,(\hat{C}_n - C) - \sqrt{n}\,(C_{\theta_n} - C_{\theta^*})$ converges weakly to the tight centred Gaussian process $\mathbb{C} - \Theta^* \nabla C_{\theta^*}$ on $\ell^\infty[0,1]^d$. When $\mathscr{H}_0$ fails, $C \neq C_{\theta^*}$ and, as $n \to \infty$, $\sqrt{n}\,\|C - C_{\theta^*}\| \to \infty$. One may then conclude. □

Proof of Proposition 3: Consider again the mapping $\psi_\theta : \mathscr{C}[0,1]^d \to \mathscr{C}[0,1]^d : g \mapsto \psi_\theta(g) = g^2/D_\theta$. It follows from the Continuous Mapping Theorem and Theorem 3 in [19] that, as $n \to \infty$,

$$\left(\psi_\theta\{\sqrt{n}(\hat{C}_n - C_{\theta_n})\}, \psi_\theta(\hat{\mathbb{C}}_n^{(1)} - \Theta^{(1)} \nabla C_{\theta_n}), \ldots, \psi_\theta(\hat{\mathbb{C}}_n^{(M)} - \Theta^{(M)} \nabla C_{\theta_n})\right)$$

converges weakly in $(\ell^\infty[0,1]^d)^{M+1}$ to

$$\left(\psi_\theta(\mathbb{C}_\theta - \Theta \nabla C_\theta), \psi_\theta(\mathbb{C}_\theta^{(1)} - \Theta^{(1)} \nabla C_\theta), \ldots, \psi_\theta(\mathbb{C}_\theta^{(M)} - \Theta^{(M)} \nabla C_\theta)\right),$$

where $(\mathbb{C}_\theta^{(1)}, \Theta^{(1)}), \ldots, (\mathbb{C}_\theta^{(M)}, \Theta^{(M)})$ are independent copies of $(\mathbb{C}_\theta, \Theta)$. Using (A1) and Proposition 4, one can proceed exactly as in the proof of Proposition 1 to complete the argument. □

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